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A numerical algorithm for a three dimensional inverse acoustic scattering problem was considered. From the knowledge of several far field patterns of the Helmholtz equation a closed surface ∂D representing the boundary of an unknown obstacle D is reconstructed. The obstacle D is supposed to be hard (i.e. Neumann boundary condition) or characterized by an acoustic impedance (i.e. mixed boundary condition).

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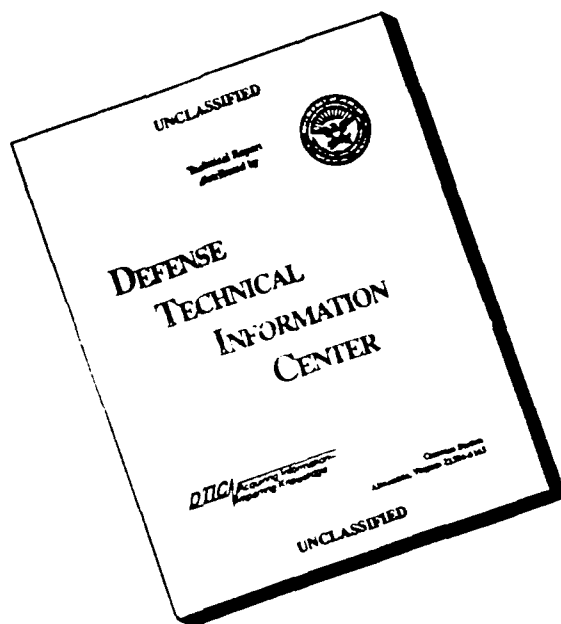
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1 Introduction

In this report we present the work performed in the period May 1990 April 1992 under contract AFOSR-90-0228: *Mathematics: Numerical solution of inverse problems in acoustics*, Principal Investigators: Luciano Misici and Francesco Zirilli.

In section 2 we present the work done in the contract, in section 3,4 and 5 three scientific papers that explain in detail the work discussed in section 2 are reproduced.

2. Statement of the work accomplished

Let \mathbb{R}^3 be the three dimensional euclidean space, $D \subset \mathbb{R}^3$ be a bounded simply connected domain that contains the origin with boundary ∂D . Let $u^i(\underline{x})$ be an "incoming" plane wave, that is:

$$u^i(\underline{x}) = e^{ik(\underline{x}, \underline{\alpha})} \quad (2.1)$$

where $k > 0$ is the wave number, α is a given unit vector and $\underline{x} = (x, y, z) \in \mathbb{R}^3$. Let $u^s(\underline{x})$ be the scattered acoustic field and $u(\underline{x})$ be the total field that is:

$$u(\underline{x}) = u^i(\underline{x}) + u^s(\underline{x}) \quad (2.2)$$

Let us consider the following "direct" problems:

Problem 2.1- Find u defined in $\mathbb{R}^3 \setminus D$ such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus D \quad (2.3)$$

$$u = 0 \quad \text{on } \partial D \quad (2.4)$$

$$\lim_{r \rightarrow \infty} r \left[\frac{\partial u^s}{\partial r} - iku^s \right] = 0 \quad (2.5)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $r = (x^2 + y^2 + z^2)^{1/2}$.

Problem 2.2- Find u defined in $\mathbb{R}^3 \setminus D$ such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus D \quad (2.6)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \quad (2.7)$$

$$\lim_{r \rightarrow \infty} r \left[\frac{\partial u^s}{\partial r} - iku^s \right] = 0 \quad (2.8)$$

where ν is the outward normal vector to ∂D .

Problem 2.3- Find u defined in $\mathbb{R}^3 \setminus D$ such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus D \quad (2.9)$$

$$\mu_1 u + \mu_2 \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \quad (2.10)$$

$$\lim_{r \rightarrow \infty} r \left[\frac{\partial u^s}{\partial r} - i k u^s \right] = 0 \quad (2.11)$$

where μ_1 and μ_2 are given constants.

Problem 2.1 and in particular the Dirichlet boundary condition (2.4) corresponds to the study of acoustically soft obstacles, Problem 2.2 and in particular the Neumann boundary condition (2.7) corresponds to the study of acoustically hard obstacles, finally Problem 2.3 and in particular the mixed boundary condition (2.10) corresponds to the study of obstacles characterized by an acoustic impedance. Moreover we have:

$$u^s(\underline{x}) = \frac{e^{ikr}}{r} F(\hat{\underline{x}}, k, \underline{\alpha}) + O\left(\frac{1}{r^2}\right) ; \quad r \rightarrow \infty \quad (2.12)$$

where $\hat{\underline{x}} = \frac{\underline{x}}{\|\underline{x}\|}$ and $F(\hat{\underline{x}}, k, \underline{\alpha})$ is the far field pattern associated to $u^s(\underline{x})$ generated by $u^i(\underline{x})$. Let λ_n , $n = 1, 2, \dots$ be the eigenvalues of the Helmholtz equation (2.3) considered in D with the appropriate boundary conditions. Let $B = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| < 1\}$ and $\partial B = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| = 1\}$, we will consider the following inverse problems:

Problem 2.4- Given $k > 0$, $-k^2 \neq \lambda_n$, $n = 1, 2, \dots$ and $\Omega_1, \Omega_2 \subseteq \partial B$ from the knowledge of the far field patterns $F(\hat{\underline{x}}, k, \underline{\alpha})$ of problem 2.1 (Dirichlet boundary value problem) for $\alpha \in \Omega_1$, $\underline{x} \in \Omega_2$ find the boundary of D , ∂D .

Problem 2.5- Is analogous to Problem 2.4 when Problem 2.2 that is the Neumann boundary condition is considered instead of Problem 2.1.

Problem 2.6- Is analogous to Problem 2.4 when Problem 2.3 that is the mixed boundary condition is considered instead of Problem 2.1.

The inverse Problems 2.4,2.5,2.6 have been considered with special attention to the resonance region case, that is the case when

$$kL \simeq 1$$

where L is a characteristic length of the obstacle.

In the research reported here we have considered two types of obstacles:

- (1) obstacles with smooth boundary ∂D . For this class of obstacles we have developed a numerical method to solve the inverse Problems 2.4,2.5,2.6 (see the references reported in sections 3,4,5 of this report). The numerical method developed generalizes the method introduced by Colton and Monk and uses the Herglotz function technique.
- (2) obstacles with Lipschitz continuous boundary. For this class of obstacles we have studied [1] the direct Problems 2.1,2.2,2.3. The existence and uniqueness of the solution of problems 2.1,2.2,2.3 has been established. In order to compute the far field patterns generated by these non smooth boundaries we have generalized the method suggested by Milder in [2]. However up to now, due to the difficulties encountered in computing the far field data, we have not been able to work out a satisfactory method to solve the inverse Problems 2.4,2.5,2.6 when non smooth boundaries are considered.

References

- [1] A. Vecchio, F. Zirilli, in preparation.
- [2] D.M. Milder: *An improved formalism for wave scattering from rough surfaces*, J. Acoust. Soc. Am., **89**, 1991, 529-541.

3 APPENDIX 1

L. Misici, F. Zirilli: "An inverse problem for the three dimensional Helmholtz equation with Neumann or mixed boundary conditions: a numerical method", in Mathematical and numerical aspects of wave propagation phenomena, G. Cohen, L. Halpern, P. Joly Editors, SIAM Proceedings Series, Philadelphia, (1991), 497-508.

An Inverse Problem for the Three Dimensional Helmholtz Equation with Neumann or Mixed Boundary Conditions: A Numerical Method*

Luciano Misicic
Francesco Zirillit

Abstract. A numerical algorithm for a three dimensional inverse acoustic scattering problem is considered. From the knowledge of several far field patterns of the Helmholtz equation a closed surface ∂D representing the boundary of an unknown obstacle D is reconstructed. The obstacle D is supposed to be hard (i.e. Neumann boundary condition) or characterized by an acoustic impedance (i.e. mixed boundary condition).

1. Introduction

Let R^3 be the three dimensional euclidean space, $\underline{x} = (x, y, z) \in R^3$ be a generic vector, $(., .)$ will denote the euclidean scalar product and $\|.\|$ the euclidean norm. Let $D \subset R^3$ be a bounded simply connected domain with smooth boundary ∂D that contains the origin. Let $u^i(\underline{x})$ be an incoming acoustic plane wave, that is:

$$u^i(\underline{x}) = e^{ik(\underline{x}, \underline{\alpha})} \quad (1.1)$$

where $k > 0$ is the wave number and $\underline{\alpha} \in R^3$ is a fixed unit vector. Let us denote with $u^s(\underline{x})$ the acoustic field scattered by the obstacle D and with $u(\underline{x})$ the total acoustic field, that is:

$$u(\underline{x}) = u^i(\underline{x}) + u^s(\underline{x}) \quad (1.2)$$

The total acoustic field $u(\underline{x})$ satisfies the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad \text{in } R^3 \setminus D \quad (1.3)$$

and the scattered acoustic field $u^s(\underline{x})$ satisfies the Sommerfeld radiation condition at infinity, that is:

$$\lim_{r \rightarrow \infty} r \left\{ \frac{\partial u^s}{\partial r} - iku^s \right\} = 0 \quad (1.4)$$

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where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the laplacian and $r = \|\underline{x}\| = (x^2 + y^2 + z^2)^{1/2}$. Moreover the total acoustic field $u(\underline{x})$ satisfies a boundary condition on ∂D . This boundary condition can be formulated in several different ways, depending on the nature of the obstacle D . In [1],[2] we have considered the acoustically soft obstacles [3] that are characterized by the Dirichlet boundary condition:

$$u = 0 \quad \text{on } \partial D \quad (1.5)$$

In this paper we restrict our attention to the acoustically hard obstacles [3] characterized by the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \quad (1.6)$$

where ν is the unit normal on ∂D , and to the obstacles characterized by an acoustic impedance χ [3] that satisfy the mixed boundary condition

$$u + \chi \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \quad (1.7)$$

We assume that χ is a given constant. We consider two boundary value problems: the Neumann boundary value problem given by (1.3),(1.4),(1.6) and the mixed boundary value problem given by (1.3),(1.4),(1.7). In [4] it is shown that the scattered field $u'(\underline{x})$ of the Neumann and mixed boundary value problem has the following expansion

$$u'(\underline{x}) = \frac{e^{ikr}}{r} F_0(\hat{\underline{x}}, k, \underline{\alpha}) + O\left(\frac{1}{r^2}\right) \quad \text{when } r \rightarrow \infty \quad (1.8)$$

where $\hat{\underline{x}} = \frac{\underline{x}}{\|\underline{x}\|}$, $\underline{x} \neq 0$ and $F_0(\hat{\underline{x}}, k, \underline{\alpha})$ is the far field pattern generated by the incoming wave (1.1) that hits the obstacle D .

In this paper we introduce a numerical method for an inverse problem for the three dimensional Helmholtz equation, that is from the knowledge of the nature of the obstacle i.e. the boundary condition on ∂D satisfied by $u(\underline{x})$ and from the far fields F_0 generated by several incoming waves we want to recover the shape of the obstacle ∂D .

The inverse acoustic scattering problem have received a lot of attention in the scientific and technical literature, here we will refer only to the work of Colton and Monk [5],[6],[7] since the work presented in this paper has been inspired by [7]. To be more precise let λ_n , $n = 1, 2, \dots$ be the eigenvalues of the Helmholtz equation in the interior of D , with Neumann boundary condition (1.6) or with the mixed boundary condition (1.7); let $B = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| < 1\}$, and ∂B be the boundary of B . We will consider the following inverse problem:

Problem 1.1 Let us assume that $u(\underline{x})$ satisfies the Neumann boundary condition (1.6) or the mixed boundary condition (1.7). Let $\Omega_1 \subseteq \partial B$, $\Omega_2 \subset \{\xi \in \mathbb{R} \mid \xi > 0\}$ be two given sets such that $\lambda_i \notin \Omega_2$ $i = 1, 2, \dots$. From the knowledge of $F_0(\hat{\underline{x}}, k, \underline{\alpha})$, for $\underline{\alpha} \in \Omega_1$, $k \in \Omega_2$ determine the boundary of the obstacle ∂D .

We note that the condition $\lambda_i \notin \Omega_2$ $i = 1, 2, \dots$ is a non-resonance condition, that Ω_1 is the set of the directions of the incoming waves and that the far field F_0 is observed for $\hat{\underline{x}} \in \partial B$.

In this paper we present a numerical method to solve Problem 1.1, in particular in section 2 we derive the mathematical relations needed to develop our method, in section 3 we present our numerical method, finally in section 4 some numerical experience is shown.

2. The mathematical formulation of the inverse problem

For $\underline{x}, \underline{y} \in \mathbb{R}^3$ let

$$\Phi(k|\underline{x} - \underline{y}|) = \frac{e^{ik|\underline{x} - \underline{y}|}}{|\underline{x} - \underline{y}|} \quad (2.1)$$

be the Green's function of the Helmholtz operator with the Sommerfeld radiation condition at infinity. It is easy to see that:

$$\Phi(k|\underline{x} - \underline{y}|) = \frac{e^{ik|\underline{x}|}}{|\underline{x}|} e^{-ik(\underline{x}, \underline{y})} + O\left(\frac{1}{|\underline{x}|^2}\right) \quad \text{when } |\underline{x}| \rightarrow \infty \quad (2.2)$$

moreover from the Helmholtz formula [8] we have:

$$\frac{1}{4\pi} \int_{\partial D} \left(u(\underline{y}) \frac{\partial \Phi}{\partial \nu(\underline{y})} - \Phi \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} \right) d\sigma(\underline{y}) = \begin{cases} -u(\underline{x}) & \text{if } \underline{x} \in D \\ u(\underline{x}) & \text{if } \underline{x} \in \mathbb{R}^3 \setminus D \end{cases} \quad (2.3)$$

where $d\sigma(\underline{y})$ is the surface measure on ∂D .

Substituting (2.2) in (2.3) and using (1.8) we have:

$$F_o(\underline{\hat{x}}, k, \underline{\alpha}) = \frac{1}{4\pi} \int_{\partial D} \left(u(\underline{y}) \frac{\partial e^{-ik(\underline{\hat{x}}, \underline{y})}}{\partial \nu(\underline{y})} - \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} e^{-ik(\underline{\hat{x}}, \underline{y})} \right) d\sigma(\underline{y}) \quad (2.4)$$

Let $g(\underline{\hat{x}}) \in L^2(\partial B, d\sigma)$ where $L^2(\partial B, d\sigma)$ is the space of square integrable functions with respect to the measure $d\sigma$ and $\overline{g(\underline{\hat{x}})}$ is the complex conjugate of $g(\underline{\hat{x}})$.

For every $g(\underline{\hat{x}}) \in L^2(\partial B, d\sigma)$ from (2.4) interchanging the integrals we have:

$$\begin{aligned} \int_{\partial B} F_o(\underline{\hat{x}}, k, \underline{\alpha}) \overline{g(\underline{\hat{x}})} d\sigma(\underline{\hat{x}}) &= \frac{1}{4\pi} \int_{\partial B} \overline{g(\underline{\hat{x}})} d\sigma(\underline{\hat{x}}) \int_{\partial D} \left(u(\underline{y}) \frac{\partial e^{-ik(\underline{\hat{x}}, \underline{y})}}{\partial \nu(\underline{y})} - \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} e^{-ik(\underline{\hat{x}}, \underline{y})} \right) d\sigma(\underline{y}) \\ &= \frac{1}{4\pi} \int_{\partial D} \left(u(\underline{y}) \frac{\partial \overline{v(k\underline{y})}}{\partial \nu(\underline{y})} - \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} \overline{v(k\underline{y})} \right) d\sigma(\underline{y}) \end{aligned} \quad (2.5)$$

where

$$v(k\underline{y}) = \int_{\partial B} g(\underline{\hat{x}}) e^{ik(\underline{\hat{x}}, \underline{y})} d\sigma(\underline{\hat{x}}) \quad (2.6)$$

It is easy to see that $v(k\underline{y})$ satisfies the Helmholtz equation for $\underline{y} \in \mathbb{R}^3$, moreover $v(k\underline{y})$ is the Herglotz wave function corresponding to the Herglotz kernel $g(\underline{\hat{x}})$. Since the total acoustic field u satisfies the boundary condition (1.6) or (1.7) on the surface ∂D formula (2.5) reduces to:

$$(i) \quad \int_{\partial B} F_o(\underline{\hat{x}}, k, \underline{\alpha}) \overline{g(\underline{\hat{x}})} d\sigma(\underline{\hat{x}}) = \frac{1}{4\pi} \int_{\partial D} u(\underline{y}) \frac{\partial \overline{v(k\underline{y})}}{\partial \nu(\underline{y})} d\sigma(\underline{y}) \quad (2.7)$$

when the Neumann boundary condition (1.6) is satisfied, or to

$$(ii) \quad \int_{\partial B} F_o(\underline{\hat{x}}, k, \underline{\alpha}) \overline{g(\underline{\hat{x}})} d\sigma(\underline{\hat{x}}) = \frac{1}{4\pi} \int_{\partial D} \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} \left[\chi \frac{\partial}{\partial \nu(\underline{y})} + 1 \right] \overline{v(k\underline{y})} d\sigma(\underline{y}) \quad (2.8)$$

when the mixed boundary condition (1.7) is satisfied.

We restrict our attention to the Neumann Herglotz domains, that is domains such that the unique solution v of:

$$(\Delta + k^2)v = 0 \quad \underline{x} \in D \quad (2.9)$$

$$\frac{\partial v}{\partial \nu} = -\frac{\partial}{\partial \nu(\underline{x})} \frac{e^{-ik\|\underline{x}\|}}{k\|\underline{x}\|} \quad \underline{x} \in \partial D \quad (2.10)$$

is given by (2.6) for a suitable choice $g_H(\underline{\hat{x}}, k)$ of $g(\underline{\hat{x}})$, or to the mixed Herglotz domains, that is domains such that the unique solution v of

$$(\Delta + k^2)v = 0 \quad \underline{x} \in D \quad (2.11)$$

$$(x \frac{\partial}{\partial \nu} + 1)v = (x \frac{\partial}{\partial \nu(\underline{x})} + 1)(-\frac{e^{-ik\|\underline{x}\|}}{k\|\underline{x}\|}) \quad \underline{x} \in \partial D \quad (2.12)$$

is given by (2.6) for a suitable choice $g_H(\underline{\hat{x}}, k)$ of $g(\underline{\hat{x}})$.

We note that in the definition of Herglotz domains we have exploited the hypothesis $k^2 \neq \lambda_i$ $i = 1, 2, \dots$. A simple computation shows that the sphere is an Herglotz domain that is the class of the Herglotz domains is not empty. In (2.7), (2.8) let v be the Herglotz wave function associated to $g_H(\underline{\hat{x}}, k)$, using (2.10), (2.12) respectively and the Helmholtz formula (2.3) we have

$$\int_{\partial B} F_o(\underline{\hat{x}}, k, \underline{\alpha}) \overline{g_H(\underline{\hat{x}}, k)} d\sigma(\underline{\hat{x}}) = \frac{1}{k} \quad (2.13)$$

when $g_H(\underline{\hat{x}}, k)$ is the Herglotz kernel, formula (2.13) holds $\forall k, \underline{\alpha}$. Problem 1.1 proposed in section 1 will be solved in three steps:

- (i) from the knowledge of some far fields F_o using (2.13) determine the Herglotz kernel $g_H(\underline{\hat{x}}, k)$ of the domain D
- (ii) from $g_H(\underline{\hat{x}}, k)$ using (2.6) find the corresponding Herglotz wave function v
- (iii) determine ∂D using (2.10) (Neumann boundary condition) or (2.12) (mixed boundary condition).

3. The numerical method

Given $D \subset \mathbb{R}^3$ and the boundary conditions (1.6) or (1.7) satisfied by u on ∂D , let $\Omega_1 = \{\underline{\alpha}_i \in \partial B \mid i = 1, 2, \dots, N\}$ be the set of directions of the incoming waves and $\Omega_2 = \{k_i > 0 \mid i = 1, 2, \dots, M\}$ be the set of wave numbers of the incoming waves. The data of our problem are the measurements of the far fields $F_o(\underline{\hat{x}}, k_j, \underline{\alpha}_i)$ $\underline{\hat{x}} \in \partial B$; $j = 1, 2, \dots, M$; $i = 1, 2, \dots, N$. In the numerical experience shown in section 4 the data are obtained by solving numerically the "direct" problems (1.3), (1.4), (1.6) or (1.3), (1.4), (1.7). Let (θ, ϕ) be the polar angles so that

$$\underline{\hat{x}}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (3.1)$$

and

$$U_{lm}(\underline{\hat{x}}) = \gamma_{lm} P_l^m(\cos \theta) \cos m\phi, \quad V_{lm}(\underline{\hat{x}}) = \gamma_{lm} P_l^m(\cos \theta) \sin m\phi$$

be the spherical harmonics that is P_l are the Legendre polynomials, P_l^m is the m^{th} derivative of P_l and γ_{lm} are the normalization factors in $L^2(\partial B, d\sigma)$. From these data our computation proceeds in four steps:

Step 1. For each j compute the Fourier coefficients of $F_o(\underline{\hat{x}}, k_j, \underline{\alpha}_i)$, $i = 1, 2, \dots, N$. Given $L_{\max} \geq 0$ we assume that the far field $F_o(\underline{\hat{x}}, k_j, \underline{\alpha}_i)$ can be approximated by a truncated Fourier series, that is:

$$F_o(\underline{\hat{x}}, k_j, \underline{\alpha}_i) = \sum_{l=0}^{L_{\max}} \sum_{m=0}^l F_{lm1}^{ij} U_{lm}(\underline{\hat{x}}) + \sum_{l=1}^{L_{\max}} \sum_{m=1}^l F_{lm2}^{ij} V_{lm}(\underline{\hat{x}}) \quad (3.2)$$

The Fourier coefficients $\{F_{lm1}^{ij}, F_{lm2}^{ij}\}$ are determined computing the following integrals:

$$F_{lm1}^{ij} = \int_{\partial B} F_0(\hat{x}, k_j, \alpha_i) U_{lm}(\hat{x}) d\sigma(\hat{x}) \quad , \quad l = 0, 1, \dots, L_{\max} \quad ; \quad m = 0, 1, \dots, l \quad (3.3)$$

$$F_{lm2}^{ij} = \int_{\partial B} F_0(\hat{x}, k_j, \alpha_i) V_{lm}(\hat{x}) d\sigma(\hat{x}) \quad , \quad l = 1, 2, \dots, L_{\max} \quad ; \quad m = 1, 2, \dots, l \quad (3.4)$$

Step 2. From the coefficients of $F_0(\hat{x}, k_j, \alpha_i)$; $i = 1, 2, \dots, N$ to the Herglotz kernels $g_H(\hat{x}, k_j)$, $j = 1, 2, \dots, M$ of the domain D .

Let $g_H(\hat{x}, k_j) \in L^2(\partial B, d\sigma)$ be the Herglotz kernel associated to the domain D when we use the wave number $k_j \in \Omega_2$. We assume for $g_H(\hat{x}, k_j)$ the expression of a truncated Fourier expansion, that is

$$g_H(\hat{x}, k_j) = \sum_{l=0}^{L_g} \sum_{m=0}^l g_{lm1}^j U_{lm}(\hat{x}) + \sum_{l=1}^{L_g} \sum_{m=1}^l g_{lm2}^j V_{lm}(\hat{x}) \quad (3.5)$$

where $0 \leq L_g \leq L_{\max}$. From (2.13) using the orthogonality properties of the spherical harmonics we have:

$$\sum_{l=0}^{L_g} \sum_{m=0}^l F_{lm1}^{ij} \overline{g_{lm1}^j} + \sum_{l=1}^{L_g} \sum_{m=1}^l F_{lm2}^{ij} \overline{g_{lm2}^j} = \frac{1}{k_j} \quad , \quad i = 1, 2, \dots, N \quad ; \quad j = 1, 2, \dots, M \quad (3.6)$$

So that for each j the Fourier coefficients $\{g_{lm1}^j, g_{lm2}^j\}$ are determined solving (3.6). For j fixed the linear system (3.6) has $\frac{(L_g+1)(L_g+2)}{2} + \frac{L_g(L_g+1)}{2}$ unknowns, that is in order to determine $\{g_{lm1}^j, g_{lm2}^j\}$ we need N incoming waves with $N \geq \frac{(L_g+1)(L_g+2)}{2} + \frac{L_g(L_g+1)}{2}$. If the obstacle D has some symmetry such as cylindrical symmetry around the z -axis and/or symmetry with respect to the equator then a similar symmetry can be assumed on $g_H(\hat{x}, k_j)$. This assumption reduces substantially the number of unknowns in (3.6) and as a consequence the number N of incoming waves needed to recover the desired approximation of $g_H(\hat{x}, k_j)$ [2].

Step 3. From the Herglotz kernels $g_H(\hat{x}, k_j)$ to the Herglotz wave functions $v_j(k, \underline{y})$. From (2.6) and (3.5) we have:

$$v_j(k, \underline{y}) = 4\pi \left\{ \sum_{l=0}^{L_g} \sum_{m=0}^l g_{lm1}^j i^l J_l(k, \|\underline{y}\|) U_{lm}(\hat{\underline{y}}) + \sum_{l=1}^{L_g} \sum_{m=1}^l g_{lm2}^j i^l J_l(k, \|\underline{y}\|) V_{lm}(\hat{\underline{y}}) \right\} \quad (3.7)$$

where $\hat{\underline{y}} = \frac{\underline{y}}{\|\underline{y}\|}$ and $J_l(\cdot)$ is the spherical Bessel function of order l .

Step 4. From the Herglotz wave functions $v_j(k, \underline{y})$; $j = 1, 2, \dots, M$ to the boundary of the obstacle ∂D .

Let (r, θ, ϕ) be the polar variables we assume that exist $0 < a < b < \infty$ and a function $f(\theta, \phi)$ with $a \leq f \leq b$ such that $\partial D = \{r = f(\theta, \phi) \mid 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}$. We approximate $f(\theta, \phi)$ with a truncated Fourier series, that is:

$$f(\theta, \phi) = \sum_{l=0}^{L_p} \sum_{m=0}^l c_{lm1} U_{lm}(\hat{x}) + \sum_{l=1}^{L_p} \sum_{m=1}^l c_{lm2} V_{lm}(\hat{x}) \quad (3.8)$$

where $L_p \geq 0$ is chosen depending on ∂D i.e. simple obstacles can be reconstructed with $L_p = 4, 6$. Moreover if ∂D has some symmetries these can be translated in properties

of the coefficients $\{c_{lm1}, c_{lm2}\}$. Let $\underline{c} = \{c_{lm1}, c_{lm2}\}$ $0 \leq l \leq L_p$, $0 \leq m \leq l$ be an $\frac{(L_p+1)(L_p+2)}{2} + \frac{L_p(L_p+1)}{2}$ dimensional vector. The unknown boundary ∂D is obtained minimizing with respect to \underline{c}

$$I_1(\underline{c}) = \sum_{j=0}^M \frac{1}{k_j} \int_0^{2\pi} d\phi \int_0^\pi \left| \frac{\partial}{\partial \nu} \left(v_j + \frac{e^{-ik_j f}}{k_j f} \right) \right|^2 \sin \theta d\theta \quad (3.9)$$

when the Neumann boundary condition (1.6) is considered, or

$$I_2(\underline{c}) = \sum_{j=0}^M \frac{1}{k_j} \int_0^{2\pi} d\phi \int_0^\pi \left| \left(\chi \frac{\partial}{\partial \nu} + 1 \right) \left(v_j + \frac{e^{-ik_j f}}{k_j f} \right) \right|^2 \sin \theta d\theta \quad (3.10)$$

when the mixed boundary condition (1.7) is considered.

In (3.9), (3.10) the Herglotz wave function v_j are computed in $(r = f(\theta, \phi), \theta, \phi)$ and ν is the unit exterior normal to the surface $r = f(\theta, \phi)$ and f is given by (3.8). The integrals in (3.9), (3.10) are approximated with some elementary quadrature formula. The weights $\frac{1}{k_j}$ are introduced in order to make the different terms in the sums over j of the same order of magnitude. When the minimization of the functions $I_i(\underline{c})$, $i = 1, 2$ does not give a satisfactory reconstruction of ∂D we minimize

$$P_i(\underline{c}) = I_i(\underline{c}) + w(\underline{c}) \quad ; \quad i = 1, 2 \quad (3.11)$$

where $w(\underline{c})$ is a penalization term.

For a large class of surfaces ∂D including the ones with cylindrical symmetry with respect to the z-axis we have $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0$ at $\theta = 0$ (North pole) and $\theta = \pi$ (South pole), moreover if ∂D is also symmetric with respect to the equator we have $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0$ at $\theta = \frac{\pi}{2}$ (the Equator).

When $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0$ the relations

$$\frac{\partial}{\partial \nu} \left(v_j + \frac{e^{-ik_j f}}{k_j f} \right) = 0 \quad (3.12)$$

and

$$\left(\chi \frac{\partial}{\partial \nu} + 1 \right) \left(v_j + \frac{e^{-ik_j f}}{k_j f} \right) = 0 \quad (3.13)$$

become nonlinear equations in the unknown f that can be solved, for example, using the bisection method. In this way we can obtain f_1^*, f_2^*, f_3^* estimates of $f(\theta, \phi)$ when $\theta = \theta_1, \theta_2, \theta_3$ with $\theta_1 = 0, \theta_2 = \pi, \theta_3 = \frac{\pi}{2}$. The penalization term $w(\underline{c})$ is given by:

$$w(\underline{c}) = \sum_{i=1}^3 p_i (f(\theta_i, \phi) - f_i^*)^2 \quad (3.14)$$

where $f(\theta, \phi)$ is given by (3.8) and $p_i \geq 0$, $i = 1, 2, 3$ are weight factors. The minimization of the functions $I_i(\underline{c})$ or $P_i(\underline{c})$, $i = 1, 2$ is performed with a stochastic global minimization algorithm introduced in [9], [10].

4. The numerical experience

In our numerical experience we chosen in (1.7) the acoustic impedance $\chi = 1$. The surfaces ∂D considered are the following ones:

- 1- Oblate Ellipsoid $(\frac{2}{3}x)^2 + (\frac{2}{3}y)^2 + z^2 = 1$ (4.1)
- 2- Prolate Ellipsoid $x^2 + y^2 + (\frac{2}{3}z)^2 = 1$ (4.2)
- 3- Short Cylinder $((\frac{2}{3}x)^2 + (\frac{2}{3}y)^2)^6 + z^{10} = 1$ (4.3)
- 4- Long Cylinder $(x^2 + y^2)^6 + (\frac{2}{3}z)^{10} = 1$ (4.4)
- 5- Vogel's Peanut $r = \frac{2}{3}(\cos^2 \theta + \frac{1}{4} \sin^2 \theta)^{1/2}$ (4.5)
- 6- Pseudo Apollo $r = \frac{3}{5}(\frac{17}{4} + 2 \cos 3\theta)^{1/2}$ (4.6)

All these surfaces are cylindrically symmetric with respect to the z -axis and the surfaces 1,2,3,4,5 are also symmetric with respect to the equator. These symmetries are always exploited to reduce the number of Fourier coefficients in the expansions of the Herglotz kernels and of the surfaces $f(\theta, \phi)$.

We observe that the obstacles D corresponding to 1,2,3,4 are convex and the ones corresponding to 5,6 are not convex. Finally a characteristic length L of the obstacles can be chosen equal 1. The set of the directions of the incoming waves is:

$$\Omega_1 = \{(\theta_j^*, 0) \mid \theta_j^* = \frac{\pi j}{N}, j = 0, 1, \dots, N\} \quad (4.7)$$

with

$$N = L_{\max} + 1 \quad (4.8)$$

The set Ω_2 is a subset of the set $\{2, 3\}$ and $L_\theta = L_{\max}$. We observe that with this choice of Ω_2 the product k, L is of order one, that is we are working in the resonance region.

For $j = 0, 1, \dots, 36$ let $\theta_j = \frac{\pi j}{36}$, $f(\theta_j, 0)$ be the exact value of the surface given by (4.1), ..., (4.6) and $f_c(\theta_j, 0)$ be the value reconstructed performing the numerical procedure described in section 3. The relative L^2 error in the points $\{\theta_j \mid j = 0, 1, \dots, 36\}$, that is

$$E_L = \left[\frac{\sum_{j=0}^{36} (f(\theta_j, 0) - f_c(\theta_j, 0))^2}{\sum_{j=0}^{36} f^2(\theta_j, 0)} \right]^{1/2} \quad (4.9)$$

is used as a performance index.

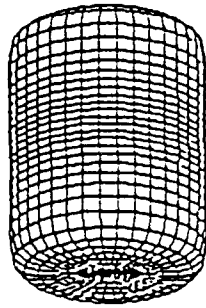
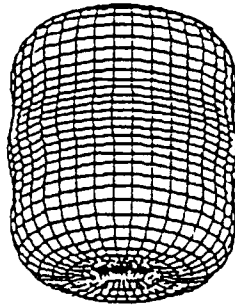
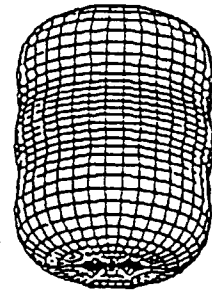
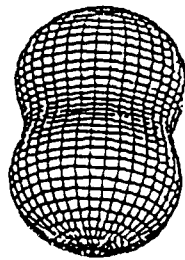
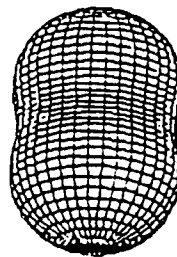
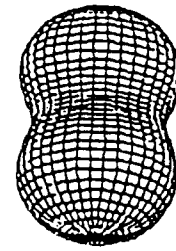
The results obtained are shown in Table 4.1, Table 4.2, Fig. 4.1a, 4.1b, 4.1c, Fig. 4.2a, 4.2b, 4.2c and Fig. 4.3a, 4.3b, 4.3c.

Table 4.1 Neumann problem

Object	Reconstruction	L_{\max}	L_θ	k_1	k_2	Penalization term		E_L
						North Pole	Equator	
Oblate Ellipsoid	1	8	4	2	no	yes	yes	0.00415
"	2	3	4	2	3	no	no	0.00412
Prolate Ellipsoid	3	3	4	2	no	yes	yes	0.00468
"	4	3	4	2	3	no	no	0.00371
Short Cylinder	5	8	6	2	no	yes	yes	0.0476
"	6	3	5	2	3	no	no	0.0361
Long Cylinder	7	3	6	2	no	yes	yes	0.0551
"	8	8	6	2	3	no	no	0.0785
Vogel's Peanut	9	8	4	2	no	yes	yes	0.0299
"	10	8	4	2	3	no	no	0.0225
Pseudo Apollo	11	8	4	2	no	yes	yes	failure
"	12	8	4	2	3	no	no	failure
"	13	8	4	2	3	yes	no	0.0353

Table 4.2 Problem with mixed boundary condition

Object	Reconstruction	L_{max}	L_p	k_1	k_2	Penalization term		E_L
						North Pole	Equator	
Oblate Ellipsoid	1	8	4	2	no	yes	yes	0.0102
"	2	8	4	2	3	no	no	0.00484
Prolate Ellipsoid	3	8	4	2	no	yes	yes	0.0139
"	4	8	4	2	3	no	no	0.0232
Short Cylinder	5	8	6	2	no	yes	yes	0.0399
"	6	8	6	2	3	no	no	0.0568
Long Cylinder	7	8	6	2	no	yes	yes	0.0424
"	8	8	6	2	3	no	no	0.0293
Vogel's Peanut	9	8	4	2	no	yes	yes	0.0448
"	10	8	4	2	3	no	no	0.0495
Pseudo Apollo	11	8	4	2	no	yes	yes	failure
"	12	8	4	2	3	no	no	failure
"	13	8	4	2	3	yes	no	0.0356

Fig. 4.1a Original
Long CylinderFig. 4.1b Reconstruction
n° 8 of table 4.1Fig. 4.1c Reconstruction
n° 8 of table 4.2Fig. 4.2a Original
Vogel's PeanutFig. 4.2b Reconstruction
n° 10 of table 4.1Fig. 4.2c Reconstruction
n° 10 of table 4.2

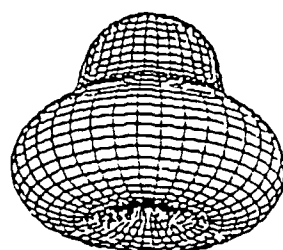


Fig. 4.3a Original
Pseudo Apollo

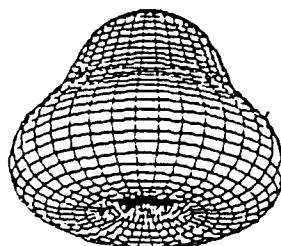


Fig. 4.3b Reconstruction
n° 13 of table 4.1

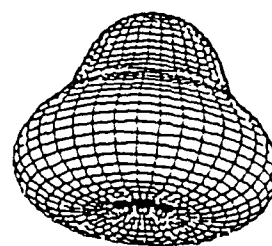


Fig 4.3c Reconstruction
n° 13 of table 4.2

The reconstruction procedure adopted here based on the minimization of the functions I_i or P_i , $i = 1, 2$ is more robust than the reconstruction procedure adopted in [1],[2] that integrates numerically an initial value problem for an ordinary differential equation. However the global minimization of the functions I_i or P_i , $i = 1, 2$ using a stochastic algorithm is computationally much more expansive than the solution of an initial value problem. The reconstruction technique suggested in this paper can be adapted to the problem with Dirichlet boundary condition with very satisfactory results.

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4 APPENDIX 2

L. Misici, F. Zirilli: "Three dimensional inverse obstacle scattering for time harmonic acoustic waves: a numerical method", submitted to SIAM J. Sci. Stat. Comp.

Three dimensional inverse obstacle scattering
for time harmonic acoustic waves: a numerical method*

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1- Introduction

Let \mathbf{R}^3 be the three dimensional euclidean space, $\underline{x} = (x, y, z)^T \in \mathbf{R}^3$ be a generic vector and the superscript T denotes the transpose operation. The euclidean scalar product will be denoted with (\cdot, \cdot) and $\|\cdot\|$ will denote the euclidean norm.

Let $D \subset \mathbf{R}^3$ be a bounded simply connected domain with smooth boundary ∂D , in the following, without loss of generality, we assume that D contains the origin. Let $u^i(\underline{x})$ be an incoming acoustic plane wave, that is:

$$u^i(\underline{x}) = e^{ik(\underline{x}, \underline{\alpha})} \quad (1.1)$$

where $k > 0$ is the wave number and $\underline{\alpha} \in \mathbf{R}^3$ is a fixed unit vector (i.e. $\|\underline{\alpha}\| = 1$). Let us denote with $u^s(\underline{x})$ the acoustic field scattered by the obstacle D and with $u(\underline{x})$ the total acoustic field, that is:

$$u(\underline{x}) = u^i(\underline{x}) + u^s(\underline{x}) \quad (1.2)$$

The total acoustic field $u(\underline{x})$ satisfies the Helmholtz equation:

$$\Delta u(\underline{x}) + k^2 u(\underline{x}) = 0 \quad \underline{x} \in \mathbf{R}^3 \setminus D \quad (1.3)$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator. The scattered field $u^s(\underline{x})$ satisfies the Sommerfeld radiation condition at infinity

$$\lim_{r \rightarrow \infty} r \left\{ \frac{\partial u^s}{\partial r} - iku^s \right\} = 0 \quad (1.4)$$

where $r = \|\underline{x}\|$. Moreover the total acoustic field $u(\underline{x})$ satisfies a boundary condition on ∂D . This boundary condition is the mathematical counterpart of the physical character of the obstacle, that is: for acoustically soft obstacles we require the Dirichlet boundary condition:

$$u(\underline{x}) = 0 \quad \underline{x} \in \partial D \quad (1.5)$$

for acoustically hard obstacles we require the Neumann boundary condition:

$$\frac{\partial u(\underline{x})}{\partial n} = 0 \quad \underline{x} \in \partial D \quad (1.6)$$

where $n(\underline{x})$, $\underline{x} \in \partial D$ is the unit outward normal to ∂D at the point \underline{x} , and finally for obstacles characterized by an acoustic impedance we require the mixed boundary condition:

$$u(\underline{x}) + \chi \frac{\partial u(\underline{x})}{\partial n} = 0 \quad \underline{x} \in \partial D \quad (1.7)$$

We assume χ to be a real constant to derive the relations of section 3, however this assumption can be avoided.

We call direct problem the problem of determining the scattered field $u^s(\underline{x})$, $\underline{x} \in \mathbb{R}^3 \setminus D$ given the incoming field $u^i(\underline{x})$ the obstacle D and its physical character, that is given one of the three boundary conditions (1.5),(1.6),(1.7). The direct problem is a boundary value problem for the Helmholtz equation, and has been widely studied. For an exposition of several mathematical results on the direct problem see for example [1]. In particular it can be shown [1] that the scattered field $u^s(\underline{x})$ corresponding to the boundary condition (1.5), (1.6) or (1.7) has the following expansion:

$$u^s(\underline{x}) = \frac{e^{ikr}}{r} F_0(\hat{\underline{x}}, k, \underline{\alpha}) + O\left(\frac{1}{r^2}\right) \quad r \rightarrow \infty \quad (1.8)$$

so that when $r \rightarrow \infty$ the leading term of the expansion in inverse powers of r is given by a spherical wave e^{ikr}/r coming out from the origin modulated by the "far field" F_0 . We note that F_0 depends on $k, \underline{\alpha}$ that are the parameters characterizing the incoming wave (1.1) and on $\hat{\underline{x}} = \frac{\underline{x}}{\|\underline{x}\|}$, for $\underline{x} \neq 0$. We note that the Helmholtz equation (1.3) is obtained from the wave equation assuming that the incoming field and the corresponding scattered field are time harmonic, that is their time dependence is given by a factor $e^{i\omega t}$ where ω is a constant.

The inverse problem that we consider here is the following: given the character of the obstacle (i.e. acoustically soft, hard, or characterized by an acoustic impedance) and the far field $F_0(\hat{\underline{x}}, k, \underline{\alpha})$ for one or several incoming waves $u^i(\underline{x})$ with different incident directions $\underline{\alpha}$ and/or wave number k determine the boundary of the obstacle ∂D . This inverse problem is known to be ill posed and due to its great interest both in mathematics and in several application fields has been widely studied, for a review see [2].

The numerical methods used to solve the inverse problem considered here can be divided in two types: the first type consists of an iterative procedure that at each step requires the numerical solution of a direct problem, the second type consists of genuine methods for the inverse problem that do not require the solution of the direct problem. In the first type we mention the work of Roger [3], Murch, Tan and Wall [4], Wang and Chen [5], Angell, Colton and Kirsch [6], Kristensson and Vogel [7]. In the second type we mention the work of Kirsch and Kress [8], [9], [10] and the work of Colton and Monk [11].

In this paper we introduce a numerical method to solve this inverse problem based on the Herglotz wave function method introduced by Colton and Monk in [11] and further developed by the authors in [12], [13], [14], [15]. In particular, based on previous work by the authors [14], we extend the

Herglotz function method introduced in [11] for acoustically soft obstacles to hard obstacles or obstacles characterized by an acoustic impedance. The analytical relations obtained are exploited to built up numerical algorithms. Finally these algorithms are efficient in the so called resonance region, that is when

$$kL = O(1) \quad (1.9)$$

where L is a characteristic length of the obstacle D . In section 2 we derive the analytical relations that are the basis of the numerical methods. In section 3 the basic numerical method developed for the solution of the inverse problem is presented. In section 4 some special features of the reconstruction procedure in the case of acoustically soft obstacles are shown. Finally in section 5 we introduce the test problems used to test the methods of sections 3 and 4 and we show the numerical results obtained.

2- The mathematical formulation of the inverse problem

For $\underline{x}, \underline{y} \in \mathbb{R}^3$ let

$$\Phi(\underline{x}, \underline{y}) = \frac{e^{ik\|\underline{x}-\underline{y}\|}}{4\pi\|\underline{x}-\underline{y}\|} \quad (2.1)$$

be the Green's function of the Helmholtz operator that satisfies the Sommerfeld radiation condition at infinity. It is easy to see that:

$$\Phi(\underline{x}, \underline{y}) = \frac{e^{ik\|\underline{x}\|}}{4\pi k\|\underline{x}\|} e^{-ik(\underline{\hat{x}}, \underline{y})} + O\left(\frac{1}{\|\underline{x}\|^2}\right) \quad \text{when } \underline{x} \rightarrow \infty \quad (2.2)$$

moreover from the Helmholtz formula [14] we have:

$$\int_{\partial D} \left[u(\underline{y}) \frac{\partial \Phi(\underline{x}, \underline{y})}{\partial n(\underline{y})} - \Phi(\underline{x}, \underline{y}) \frac{\partial u(\underline{y})}{\partial n(\underline{y})} \right] d\sigma(\underline{y}) = \begin{cases} -u^i(\underline{x}) & \text{if } \underline{x} \in D \\ u^s(\underline{x}) & \text{if } \underline{x} \in \mathbb{R}^3 \setminus D \end{cases} \quad (2.3)$$

where $d\sigma(\underline{y})$ is the surface measure on ∂D . Substituting (2.2) in (2.3) and using (1.8) we have:

$$F_0(\underline{\hat{x}}, k, \alpha) = \frac{1}{4\pi} \int_{\partial D} \left[u(\underline{y}) \frac{\partial(e^{-ik(\underline{\hat{x}}, \underline{y})})}{\partial n(\underline{y})} - \frac{\partial u(\underline{y})}{\partial n(\underline{y})} e^{-ik(\underline{\hat{x}}, \underline{y})} \right] d\sigma(\underline{y}) \quad (2.4)$$

Let $B = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| \leq 1\}$, ∂B be the boundary of B and $d\lambda$ be the surface measure on ∂B , we denote with $L^2(\partial B, d\lambda)$ the space of square integrable complex functions with respect to the measure $d\lambda$. For $g(\underline{\hat{x}}) \in L^2(\partial B, d\lambda)$ and (2.4) interchanging the integrals we have:

$$\begin{aligned} \int_{\partial B} F_0(\underline{\hat{x}}, k, \alpha) \overline{g(\underline{\hat{x}})} d\lambda(\underline{\hat{x}}) &= \frac{1}{4\pi} \int_{\partial B} \overline{g(\underline{\hat{x}})} d\lambda(\underline{\hat{x}}) \int_{\partial D} \left[u(\underline{y}) \frac{\partial(e^{-ik(\underline{\hat{x}}, \underline{y})})}{\partial n(\underline{y})} - \frac{\partial u(\underline{y})}{\partial n(\underline{y})} e^{-ik(\underline{\hat{x}}, \underline{y})} \right] d\sigma(\underline{y}) = \\ &= \frac{1}{4\pi} \int_{\partial D} \left[u(\underline{y}) \frac{\partial \overline{v(\underline{y})}}{\partial n(\underline{y})} - \frac{\partial u(\underline{y})}{\partial n(\underline{y})} \overline{v(\underline{y})} \right] d\sigma(\underline{y}) \end{aligned} \quad (2.5)$$

where

$$v(\underline{y}) = \int_{\partial B} g(\underline{\hat{x}}) e^{ik(\underline{\hat{x}}, \underline{y})} d\lambda(\underline{\hat{x}}) \quad (2.6)$$

It is easy to see differentiating under the integral sign that $v(\underline{y})$ is a solution of the Helmholtz equation for every $\underline{y} \in \mathbb{R}^3$.

Let Σ_1 , Σ_2 , Σ_3 be the sets of the eigenvalues of the Laplace operator inside the domain D with the Dirichlet boundary condition (1.5), the Neumann boundary condition (1.6) or the mixed boundary condition (1.7) respectively. We give the following definitions:

Definition 2.1 Given $-k^2 \notin \Sigma_1$, let $w_1(\underline{y})$ be the unique solution of the following boundary value problem:

$$(\Delta + k^2)w_1(\underline{y}) = 0 \quad \underline{y} \in D \quad (2.7)$$

$$w_1(\underline{y}) = -\frac{e^{-ik\|\underline{y}\|}}{k\|\underline{y}\|} \quad \underline{y} \in \partial D \quad (2.8)$$

We say that D is an Herglotz domain with respect to the Dirichlet boundary condition if there exists $g_1(\underline{\hat{x}}) \in L^2(\partial B, d\lambda)$ such that

$$w_1(\underline{y}) = \int_{\partial B} g_1(\underline{\hat{x}}) e^{ik(\underline{\hat{x}}, \underline{y})} d\lambda(\underline{\hat{x}}) \quad (2.9)$$

In a similar way we define:

Definition 2.2 Given $-k^2 \notin \Sigma_2$, let $w_2(\underline{y})$ be the unique solution of the following boundary value problem:

$$(\Delta + k^2)w_2(\underline{y}) = 0 \quad \underline{y} \in D \quad (2.10)$$

$$\frac{\partial w_2(\underline{y})}{\partial n(\underline{y})} = -\frac{\partial}{\partial n(\underline{y})} \left(\frac{e^{-ik\|\underline{y}\|}}{k\|\underline{y}\|} \right) \quad \underline{y} \in \partial D \quad (2.11)$$

We say that D is an Herglotz domain with respect to the Neumann boundary condition if there exists $g_2(\underline{\hat{x}}) \in L^2(\partial B, d\lambda)$ such that

$$w_2(\underline{y}) = \int_{\partial B} g_2(\underline{\hat{x}}) e^{ik(\underline{\hat{x}}, \underline{y})} d\lambda(\underline{\hat{x}}) \quad (2.12)$$

Definition 2.3 Given $\chi \in \mathbb{R}$ and $-k^2 \notin \Sigma_3$, let $w_3(\underline{y})$ be the unique solution of the following boundary value problem:

$$(\Delta + k^2)w_3(\underline{y}) = 0 \quad \underline{y} \in D \quad (2.13)$$

$$w_3(\underline{y}) + \chi \frac{\partial w_3(\underline{y})}{\partial n(\underline{y})} = -(1 + \chi \frac{\partial}{\partial n(\underline{y})}) \left(\frac{e^{-ik\|\underline{y}\|}}{k\|\underline{y}\|} \right) \quad \underline{y} \in \partial D \quad (2.14)$$

We say that D is an Herglotz domain with respect to the mixed boundary condition if there exists $g_3(\underline{x}) \in L^2(\partial B, d\lambda)$ such that

$$w_3(\underline{y}) = \int_{\partial B} g_3(\underline{x}) e^{ik(\underline{x}, \underline{y})} d\lambda(\underline{x}) \quad (2.15)$$

To our knowledge it is not known a characterization of the Herglotz domains, however it is easy to see that the class of Herglotz domains is not empty. In fact a straightforward computation shows that the sphere of center the origin is an Herglotz domain in the sense of Definition 2.1, 2.2, 2.3, moreover the numerical experience of section 5 can be regarded as experimental evidence that the domains considered satisfy the previous definitions.

Since now on we consider only domains D that satisfy the appropriate Herglotz condition that is Definition 2.1 or 2.2 or 2.3. In the case of the inverse problem for the acoustically soft obstacle from (2.5) using (1.5) and (2.8) we have:

$$\int_{\partial B} F_0(\underline{x}, k, \underline{\alpha}) \overline{g_1(\underline{x})} d\lambda(\underline{x}) = \frac{1}{k} \quad \forall \underline{\alpha} \in \partial B \quad (2.16)$$

Reasoning in the same way we have:

$$\int_{\partial B} F_0(\underline{x}, k, \underline{\alpha}) \overline{g_2(\underline{x})} d\lambda(\underline{x}) = \frac{1}{k} \quad \forall \underline{\alpha} \in \partial B \quad (2.17)$$

for the acoustically hard obstacle and

$$\int_{\partial B} F_0(\underline{x}, k, \underline{\alpha}) \overline{g_3(\underline{x})} d\lambda(\underline{x}) = \frac{1}{k} \quad \forall \underline{\alpha} \in \partial B \quad (2.18)$$

for the obstacle characterized by an acoustic impedance.

The numerical method for the inverse problem for the acoustically soft obstacle is based on the relations (2.16), (2.9), (2.8) that connect the data that is the far fields to the unknown ∂D . In a similar way we will exploit (2.17), (2.12), (2.11) to solve the inverse problem for the acoustically hard obstacle and (2.18), (2.15), (2.14) to solve the inverse problem for the obstacle characterized by an acoustic impedance.

3- The numerical method

Given $D \subset \mathbb{R}^3$ and the boundary condition (1.5) or (1.6) or (1.7) satisfied by u on ∂D we will exploit numerically the analytic relations derived in section 2 as follows. Since most of the content of this section is independent of the boundary conditions chosen, in order to fix the ideas, we consider

the inverse problem for acoustically soft obstacles that is the relations (2.16), (2.9), (2.3), when necessary we will comment on the peculiar features of the corresponding problems for acoustically hard obstacles or obstacles characterized by an acoustic impedance. Our general strategy can be summarized in three points:

- (i) use (2.16) to go from the knowledge of the far fields F_0 to the Herglotz kernel $g_1(\underline{\hat{x}})$ of the domain D
- (ii) use (2.9) to go from the knowledge of the Herglotz kernel $g_1(\underline{\hat{x}})$ to the Herglotz wave function $w_1(\underline{y})$
- (iii) use (2.8) to go from the knowledge of the Herglotz wave function $w_1(\underline{y})$ to the boundary of the obstacle ∂D

More precisely given D , let $\Omega_1 = \{\underline{\alpha}_i \in \partial B \mid i = 1, 2, \dots, N\}$ be the set of directions of the incoming waves, $\Omega_2 = \{k_i \in \mathbf{R} \mid -k_i^2 \notin \Sigma_1, i = 1, 2, \dots, N_1\}$ be the set of the non-resonant wave numbers of the incoming waves and $\Omega_3 = \{\underline{\hat{x}}_i \in \partial B \mid i = 1, 2, \dots, M\}$ be the set of directions where the far fields F_0 are measured. For $i = 1, 2, 3$ we assume that the elements of Ω_i are distinct. The data of our inverse problems will be the numbers $F_{i,i_1,j}$, $i = 1, 2, \dots, N$, $i_1 = 1, 2, \dots, N_1$, $j = 1, 2, \dots, M$ that represent the measurements of $F_0(\underline{\hat{x}}_j, k_{i_1}, \underline{\alpha}_i)$. In the numerical experience shown in section 5 these data are obtained solving numerically the direct problem (1.3), (1.4), (1.5).

Let (θ, ϕ) be the polar angles so that

$$\underline{\hat{x}} = \underline{\hat{x}}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T \quad (3.1)$$

and $U_{lm}(\underline{\hat{x}}) = \gamma_{lm} P_l^m(\cos \theta) \cos m\phi$, $V_{lm}(\underline{\hat{x}}) = \gamma_{lm} P_l^m(\cos \theta) \sin m\phi$, $l = 0, 1, 2, \dots$, $m = 0, 1, \dots, l$ be the spherical harmonics, where γ_{lm} are the normalization factors in $L^2(\partial B, d\lambda)$ and P_l^m are the Legendre functions. It is well known that $\{U_{lm}, V_{lm}\}_{\substack{l=0,1,\dots \\ m=0,1,\dots,l}}$ is an orthonormal complete set of $L^2(\partial B, d\lambda)$.

From the data our computation proceeds as follows:

Step 1. From the data to the coefficients of the expansion in spherical harmonics of the far field $F_0(\underline{\hat{x}}, k_{i_1}, \underline{\alpha}_i)$, $i_1 = 1, 2, \dots, N_1$, $i = 1, 2, \dots, N$.

Let us first consider the case $\Omega_3 = \{\underline{\hat{x}}_i \mid i = 1, 2, \dots, M\}$. Given an integer $L_{max} \geq 0$ we assume that the far field $F_0(\underline{\hat{x}}, k_{i_1}, \underline{\alpha}_i)$ can be approximated by a truncated expansion in spherical harmonics, that is:

$$F_0(\underline{\hat{x}}, k_{i_1}, \underline{\alpha}_i) = \sum_{l=0}^{L_{max}} \sum_{m=0}^l F_{0,l,m,1}^{i_1,i} U_{lm}(\underline{\hat{x}}) + \sum_{l=1}^{L_{max}} \sum_{m=1}^l F_{0,l,m,2}^{i_1,i} V_{lm}(\underline{\hat{x}}) \quad (3.2)$$

we note that the unknown coefficients appearing in the truncated expansion (3.2) are $(L_{max} + 1)^2$ complex constants. To determine these coefficients we impose that

$$\sum_{l=0}^{L_{max}} \sum_{m=0}^l F_{0,l,m,1}^{i_1,i} U_{lm}(\hat{x}_j) + \sum_{l=1}^{L_{max}} \sum_{m=1}^l F_{0,l,m,2}^{i_1,i} V_{lm}(\hat{x}_j) = F_{i,i_1,j} \quad , \quad j = 1, 2, \dots, M \quad (3.3)$$

The equations (3.3) are a linear system of M equations in $(L_{max} + 1)^2$ unknowns, so that in order to determine the unknowns we need $M \geq (L_{max} + 1)^2$. The condition number of the linear system (3.3) grows dramatically with the number of equations M , so that its numerical solution become practically unfeasible even for small values of L_{max} such as $L_{max} = 4, 6, 8$.

The use of regularization procedures such as Tichonov regularization to solve (3.3) are helpful but insufficient to cure the ill conditioning. We note that the right hand side of (3.3) is supposed to represent actual physical measurements that are affected by significant experimental errors so that ill conditioning is a true challenge. This ill conditioning problem is solved reducing the number of unknowns to be determined and reducing in a similar way the number of equations to be solved. This is obtained exploiting the special features of the linear system (3.3).

Step 2. From the far fields $F_0(\hat{x}, k_{i_1}, \alpha_i)$ $i_1 = 1, 2, \dots, N_1$ $i = 1, 2, \dots, N$ to the Herglotz kernel $g_{i_1}(\hat{x})$

Let $g_{i_1}(\hat{x}) \in L^2(\partial B, d\lambda)$ be the Herglotz kernel associated to the boundary condition (1.5), the domain D and the wave number $k_{i_1} \in \Omega_2$. We assume that $g_{i_1}(\hat{x})$ can be approximated by a truncated expansion in spherical harmonics, that is:

$$g_{i_1}(\hat{x}) = \sum_{l=0}^{L_g} \sum_{m=0}^l g_{lm1}^{i_1} U_{lm}(\hat{x}) + \sum_{l=1}^{L_g} \sum_{m=1}^l g_{lm2}^{i_1} V_{lm}(\hat{x}) \quad (3.4)$$

where $0 \leq L_g \leq L_{max}$ is an integer. Using the orthogonality properties of the spherical harmonics the relation (2.16) will be approximated with:

$$\sum_{l=0}^{L_g} \sum_{m=0}^l F_{0,l,m,1}^{i_1,i} \overline{g_{lm1}^{i_1}} + \sum_{l=1}^{L_g} \sum_{m=1}^l F_{0,l,m,2}^{i_1,i} \overline{g_{lm2}^{i_1}} = \frac{1}{k_{i_1}} \quad , \quad i = 1, 2, \dots, N \quad (3.5)$$

The equations (3.5) are a linear system of N equations in the $(L_g + 1)^2$ unknown coefficients $\{g_{lm1}^{i_1}, g_{lm2}^{i_1}\}$, so that in order to determine the unknowns we need $N \geq (L_g + 1)^2$. We remark that the unknown coefficients $\{g_{lm1}^{i_1}, g_{lm2}^{i_1}\}$ are complex numbers. The number of incoming waves N can be drastically reduced if the unknown coefficients to be determined are reduced in a corresponding manner. This reduction can be achieved if we make some a priori assumptions about the symmetries

of the obstacle D , and we assume than the same symmetry is conserved in the Herglotz kernel $g_{i_1}(\underline{\hat{x}})$.

In this paper we consider only two choices:

- (i) ∂D is cylindrically symmetric with respect the z-axis, that is $\partial D = \{\underline{x} = f(\theta)\underline{\hat{x}}(\theta, \phi) \mid 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}$ for some function f . In this case if we assume the same symmetry for $g_{i_1}(\underline{\hat{x}})$ we have $g_{lm1}^{i_1} = 0$ if $m > 0$ and $g_{lm2}^{i_1} = 0$ for every l, m so that the linear system (3.5) has only $L_g + 1$ unknowns and as a consequence we need only $N \geq L_g + 1$ incoming waves
- (ii) ∂D is cilindrically symmetric with respect to the z-axis and symmetric with respect to the equator. Arguing as in (i) we can conclude that the linear system (3.5) has only $\left[\frac{L_g}{2}\right] + 1$ unknowns. With $\left[\frac{L_g}{2}\right]$ we have denoted the integer part of $\frac{L_g}{2}$.

When it is necessary the system (3.5) is solved using a regularization procedure. We note that when it is assumed some symmetry about ∂D and these symmetries are exploited as previously shown a large number of the coefficients $\{F_{0,l,m,1}^{i_1,i}, F_{0,l,m,2}^{i_1,i}\}$ of the system (3.5) determined in Step 1 are multiplied by unknowns $\{g_{lm1}^{i_1}, g_{lm2}^{i_1}\}$ that are assumed zero.

Step 3. From the Herglotz kernel $g_{i_1}(\underline{\hat{x}})$ to the Herglotz wave function $w_{1,i_1}(\underline{y})$, $i_1 = 1, 2, \dots, N_1$
From (2.9) and (3.4) an explicit computation gives us an approximated expression of $w_{1,i_1}(\underline{y})$, that is:

$$w_{1,i_1}(\underline{y}) = 4\pi \left\{ \sum_{l=0}^{L_g} \sum_{m=0}^l g_{lm1}^{i_1} i^l j_l(k_{i_1} \|\underline{y}\|) U_{lm}(\underline{\hat{y}}) + \sum_{l=1}^{L_g} \sum_{m=1}^l g_{lm2}^{i_1} i^l j_l(k_{i_1} \|\underline{y}\|) V_{lm}(\underline{\hat{y}}) \right\} \quad (3.6)$$

where $\underline{\hat{y}} = \underline{y}/\|\underline{y}\|$ and j_l is the spherical Bessel function of order l .

Step 4. From the Herglotz functions $w_{1,i_1}(\underline{y})$, $i_1 = 1, 2, \dots, N_1$ to the boundary of the obstacle ∂D .

For simplicity we assume that there exists $0 < a < b < +\infty$ and a smooth function $f(\theta, \phi)$ such that $a < f(\theta, \phi) < b$, $\forall \theta, \phi$ and we have $\partial D = \{(r, \theta, \phi) \mid r = f(\theta, \phi), 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}$.

The reconstruction of the boundary ∂D from the Herglotz functions is based on the relation (2.8) in the case of the acoustically soft obstacle, or the relation (2.11) in the case of the acoustically hard obstacle, or the relation (2.14) in the case of the obstacle characterized by an acoustic impedance.

The relation (2.8) must be interpreted as a nonlinear equation that defines implicitly f as a function of θ and ϕ , the relations (2.11), (2.14), due to the presence of the $\partial/\partial n$ term, are nonlinear expressions involving $f, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}$ that is first order partial differential equations. So that we expect that the reconstruction of ∂D in the acoustically soft case should be easier than in the remaining cases. In section 4 we will show an ad hoc procedure to exploit (2.8), here we restrict our attention

to a general procedure to obtain ∂D that can be applied always. We approximate ∂D with a truncated series of spherical harmonics:

$$f(\theta, \phi) = \sum_{l=0}^{L_p} \sum_{m=0}^l c_{lm1} U_{lm}(\hat{x}) + \sum_{l=1}^{L_p} \sum_{m=1}^l c_{lm2} V_{lm}(\hat{x}) \quad (3.7)$$

where $L_p \geq 0$ is chosen depending on ∂D , i.e. simple obstacles can be reconstructed with small L_p , that is $L_p = 4, 6$. Moreover if it is assumed that ∂D has some of the symmetries previously considered these symmetries can be easily translated into properties of the coefficients $\{c_{lm1}, c_{lm2}\}$, that is the appropriate coefficients can be chosen zero. Let $\underline{c} = \{c_{lm1}, c_{lm2}\}$, $0 \leq l \leq L_p$, $0 \leq m \leq l$ be the $(L_p + 1)^2$ dimensional real vector of the unknown coefficients, the unknown boundary is obtained minimizing with respect to \underline{c} the following functions

$$I_1(\underline{c}) = \sum_{i_1=1}^{N_1} \frac{1}{k_{i_1}} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \left| w_{1,i_1} + \frac{e^{-ik_{i_1}f}}{k_{i_1}f} \right|^2 d\theta \quad (3.8)$$

when the acoustically soft obstacle is considered,

$$I_2(\underline{c}) = \sum_{i_1=1}^{N_1} \frac{1}{k_{i_1}} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \left| \frac{\partial}{\partial n} (w_{2,i_1} + \frac{e^{-ik_{i_1}f}}{k_{i_1}f}) \right|^2 d\theta \quad (3.9)$$

when the acoustically hard obstacle is considered,

$$I_3(\underline{c}) = \sum_{i_1=1}^{N_1} \frac{1}{k_{i_1}} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \left| (1 + \chi \frac{\partial}{\partial n}) (w_{3,i_1} + \frac{e^{-ik_{i_1}f}}{k_{i_1}f}) \right|^2 d\theta \quad (3.10)$$

when the obstacle with acoustic impedance is considered.

In (3.8), (3.9), (3.10) the Herglotz wave functions obtained in Step 3 are computed in $(r = f(\theta, \phi), \theta, \phi)$, $\frac{\partial}{\partial n}$ is the normal derivative with respect to the surface $r = f(\theta, \phi)$ and f is approximated with (3.7). The factor $\frac{1}{k_{i_1}}$ in (3.8), (3.9), (3.10) is chosen to make the addenda corresponding to different values of k of the same order of magnitude, finally the integrals appearing in (3.8), (3.9), (3.10) are approximated by some simple quadrature rule. We observe that the functions $I_i(\underline{c})$, $i = 1, 2, 3$ are non negative functions and that, if we neglect the effects of the approximations introduced, the surface $r = f(\theta, \phi)$ corresponds to a point where $I_i(\underline{c})$ $i = 1, 2, 3$ is zero, that is $r = f(\theta, \phi)$ is a global minimizer of $I_i(\underline{c})$. We note that in general the surface ∂D is only a proper subset of the set of points satisfying the relations (2.8) or (2.11) or (2.14).

When the minimization of $I_i(\underline{c})$ does not give a satisfactory reconstruction of ∂D we minimize $P_i(\underline{c})$ instead of $I_i(\underline{c})$, where:

$$P_i(\underline{c}) = I_i(\underline{c}) + q(\underline{c}) \quad i = 1, 2, 3 \quad (3.11)$$

and $q(\underline{c})$ is a penalization term. The penalization term $q(\underline{c})$ is chosen as follows

$$q(\underline{c}) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} p_{ij} (f(\theta_i, \phi_j) - f_{ij}^*)^2 \quad (3.12)$$

where $p_{ij} > 0$ are weight factors, f_{ij}^* are approximated values of f in the direction (θ_i, ϕ_j) that we assume known. For the acoustically soft obstacles the values f_{ij}^* can be obtained solving (2.8) with $\theta = \theta_i$, $\phi = \phi_j$ as a nonlinear equation for f . In the remaining cases the nonlinear partial differential equations (2.11), (2.14) become nonlinear equations when

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \phi} = 0 \quad (3.13)$$

For a large class of surfaces (3.13) is satisfied in some special points such as the North Pole (i.e. $\theta = 0$, ϕ arbitrary), the South Pole (i.e. $\theta = \pi$, ϕ arbitrary) or the Equator (i.e. $\theta = \frac{\pi}{2}$, ϕ arbitrary). At these points (2.11), (2.14) can be solved as nonlinear equations to obtain approximate values f_{ij}^* of f to be used in the penalization term (3.12).

The minimization of $I_i(\underline{c})$ or $P_i(\underline{c})$ is performed with one of the following three algorithms: DB-CONF [17], that is a quasi Newton local minimization algorithm, SIGMA [18],[19], that is a global minimization stochastic algorithm, or DUNLSJ [20], that is a non linear least squares algorithm. The quasi-Newton algorithm performs only a local minimization but is computationally cheaper than the global minimization algorithm. The nonlinear least squares algorithm is specifically suited for the minimization of $I_i(\underline{c})$ $i = 1, 2, 3$.

4- The reconstruction of ∂D in the acoustically soft case

In section 3 we have observed that (2.8) can be interpreted as a nonlinear equation defining f implicitly as a function of θ and ϕ . So that differentiating (2.8) with respect to θ and ϕ we have:

$$\left[k_{i_1} \frac{\partial w_{1,i_1}}{\partial r} + \left(if + \frac{1}{k_{i_1}} \right) \frac{e^{-ik_{i_1}f}}{f^2} \right] \frac{\partial f}{\partial \theta} + \frac{\partial w_{1,i_1}}{\partial \theta} = 0 \quad , \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi \quad (4.1)$$

$$\left[k_{i_1} \frac{\partial w_{1,i_1}}{\partial r} + \left(if + \frac{1}{k_{i_1}} \right) \frac{e^{-ik_{i_1}f}}{f^2} \right] \frac{\partial f}{\partial \phi} + \frac{\partial w_{1,i_1}}{\partial \phi} = 0 \quad , \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi \quad (4.2)$$

Equations (4.1), (4.2) can be interpreted as ordinary differential equations for the unknown f . We note that, in the reconstruction procedure considered here, f is not approximated with a truncated series in spherical harmonics as in (3.7) and that the choice of several values of k (i.e. k_{i_1} , $i_1 = 1, 2, \dots, N_1$) does not play any role. In order to obtain $f(\theta, \phi)$ we proceed as follows:

- (i) for fixed θ and ϕ , let say $\theta = \phi = 0$, we solve (2.8) as a nonlinear equation. Let f_{00} be the solution found
- (ii) given f_{00} we solve, for $0 \leq \theta \leq \pi$, the differential equation obtained by taking the real part of (4.1) with the initial condition

$$f(0, 0) = f_{00} \quad (4.3)$$

Let $f_0(\theta, 0)$ be the solution found. We verify that $f_0(\theta, 0)$ satisfies (2.8)

- (iii) given $f_0(\theta, 0)$ we solve, for $0 \leq \phi < 2\pi$, the differential equation obtained by taking the real part of (4.2) with the initial condition

$$f(\theta, 0) = f_0(\theta, 0) \quad (4.4)$$

Finally we verify that the $f(\theta, \phi)$ obtained in this way satisfies (2.8).

The differential problems considered in (ii), (iii) are initial value problems for scalar differential equations and are solved numerically using a Runge-Kutta-Fehlberg method (i.e. the subroutine RKF45 [21]).

The problem considered in (iii) is performed only a finite number of times corresponding to a discretization of the variable θ .

We note that if the obstacle is cylindrically symmetric with respect to the z axis (iii) is not necessary and the problem is solved after performing (i), (ii). Moreover the differential equations of (iii) for different values of θ are independent one from the other and can be solved in parallel. The set of solutions of (2.8) in general contains the surface $f(\theta, \phi)$ as a subset so that when performing (i), (ii), (iii) only trajectories that appear to define a closed surface should be considered. Finally the solution of the inverse problem based on the differential equations (4.1), (4.2) is limited to the acoustically soft obstacles and appears to be more sensible to error in the data than the minimization procedure described in section 3. However its computational cost is very small compared to the minimization procedures previously described.

5. The numerical experience

In this section we describe the numerical results obtained using the numerical methods described in sections 3 and 4 on several test problems. When we consider the boundary condition (1.7) we choose $\chi = 1$. The surfaces ∂D considered are the following ones:

$$\left(\frac{2}{3}x\right)^2 + \left(\frac{2}{3}y\right)^2 + z^2 = 1 \quad \text{Oblate Ellipsoid} \quad (5.1)$$

$$x^2 + y^2 + \left(\frac{2}{3}z\right)^2 = 1 \quad \text{Prolate Ellipsoid} \quad (5.2)$$

$$\left(\left(\frac{2}{3}x\right)^2 + \left(\frac{2}{3}y\right)^2\right)^5 + z^{10} = 1 \quad \text{Short Cylinder} \quad (5.3)$$

$$(x^2 + y^2)^5 + \left(\frac{2}{3}z\right)^{10} = 1 \quad \text{Long Cylinder} \quad (5.4)$$

$$r = \frac{2}{3}(\cos^2 \theta + \frac{1}{4} \sin^2 \theta)^{\frac{1}{2}} \quad \text{Vogel's Peanut} \quad (5.5)$$

$$r = 1 - \frac{1}{2} \cos 2\theta \quad \text{Horizontal Platelet} \quad (5.6)$$

$$r = 1 + \frac{1}{2} \cos 2\theta \quad \text{Vertical Peanut} \quad (5.7)$$

$$r = \frac{3}{5}\left(\frac{17}{4} + 2 \cos 3\theta\right)^{\frac{1}{2}} \quad \text{Pseudo Apollo} \quad (5.8)$$

$$r = \left(\left(\frac{\sin \theta}{H(\phi)}\right)^2 + \left(\frac{2}{3} \cos \theta\right)^2\right)^{-\frac{1}{2}} \quad \text{Corrugated Ellipsoid} \quad (5.9)$$

$$r = \left(\left(\frac{\sin \theta}{h(\phi)}\right)^2 + \left(\frac{2}{3} \cos \theta\right)^2\right)^{-\frac{1}{2}} \quad \text{Ellipsoid} \quad (5.10)$$

$$r = \left(\left(\frac{2 \sin \theta}{3H(\phi)}\right)^{10} + \cos^{10} \theta\right)^{-\frac{1}{10}} \quad \text{Corrugated Cylinder} \quad (5.11)$$

$$r = \frac{1}{2} + H(\phi) \sin^2 \theta \quad \text{Corrugated Platelet} \quad (5.12)$$

where

$$h(\phi) = \left(\left(\frac{3}{4} \cos \phi\right)^2 + \sin^2 \phi\right)^{-\frac{1}{2}} \quad (5.13)$$

$$H(\phi) = (R_h + A_h \cos 4\phi + B_h \cos 8\phi + C_h \cos 16\phi)^2 \quad (5.14)$$

and

$$A_h = \frac{0.3}{1.34} f_h ; \quad B_h = \frac{0.05}{1.34} f_h ; \quad C_h = -\frac{0.01}{1.34} f_h ; \quad R_h = 1 - (A_h + B_h + C_h) \quad (5.15)$$

f_h is the corrugation parameter. In our numerical experience $f_h = 0.2$. The obstacles D corresponding to (5.1), (5.2), (5.3), (5.4) are convex bodies symmetric with respect to the z -axis and to the equator, the obstacles corresponding to (5.5), (5.6), (5.7), (5.8) are non convex but they maintain the symmetry with respect to the z -axis. Finally the obstacles D corresponding to (5.9), (5.10), (5.11), (5.12) in general are non convex and non symmetric with respect to the z -axis. We observe that a characteristic length L of the obstacles can be chosen to be one. The Tables 5.1, 5.2, 5.3, 5.4 show some numerical results obtained with the methods described in the previous sections. In the E_{L^2} and E_{max} columns of those tables we use the notation $\alpha(\beta)$ to mean $\alpha \cdot 10^\beta$. Table 5.1 summarizes the results obtained with the obstacles (5.1), (5.2), (5.3), (5.4), Table 5.2 summarizes the results obtained with the obstacles (5.5), (5.6), (5.7), (5.8), finally Table 5.3 summarizes the results obtained with the obstacles (5.9), (5.10), (5.11), (5.12).

We remark that in the reconstructions presented in Table 5.3 the obstacles are considered as general surfaces, that is there is no use of symmetries in the reconstruction procedure. In Figs. 5.1, 5.2,...,5.9 the reconstructions denoted with (*) in the Tables 5.1, 5.2, 5.3 are shown.

Let $(\theta_i, \phi_j) = (i\frac{\pi}{36}, j\frac{\pi}{18})$, $i = 1, 2, \dots, 35$, $j = 0, 1, \dots, 35$ and let $f(\theta_i, \phi_j)$ be the exact values of the surfaces given by (5.1), (5.2), ..., (5.12) and $f_c(\theta_i, \phi_j)$ be the corresponding values obtained by the reconstruction procedure of sections 3 and 4. In the Tables 5.1, 5.2, 5.3, 5.4 we use as performance index the relative L^2 error, that is:

$$E_{L^2} = \frac{\left[(f(0,0) - f_c(0,0))^2 + (f(\pi,0) - f_c(\pi,0))^2 + \sum_{i=1}^{35} \sum_{j=0}^{35} (f(\theta_i, \phi_j) - f_c(\theta_i, \phi_j))^2 \right]^{1/2}}{\left[f(0,0)^2 + f(\pi,0)^2 + \sum_{i=1}^{35} \sum_{j=0}^{35} f(\theta_i, \phi_j)^2 \right]^{1/2}} \quad (5.16)$$

Table 5.1 Axially symmetric convex obstacles

$$L_{max} = L_g = 8 ; \Omega_1 = A_1 ; \Omega_3 = A_2$$

Object	Reconstruction	Boundary condition	L, k_1, k_2			Reconstruction method	Penalization term		ϵ	E_{L^2}
			L	k_1	k_2		North Pole	Equator		
Oblate Ellipsoid	1	Dirichlet	-	3	-	RKF45	-	-	0.0	0.868(-4)
"	2	"	-	3	-	RKF45	-	-	0.05	0.703(-3)
"	3(*)	Neumann	4	2	no	DBCONF	no	no	0.0	0.546(-2)
"	4(*)	"	4	2	no	DBCONF	no	no	0.05	0.880(-2)
"	5	Mixed	4	2	no	DBCONF	no	no	0.0	0.365(-2)
"	6	"	4	2	no	DBCONF	no	no	0.05	0.589(-2)
Prolate Ellipsoid	7	Dirichlet	-	3	-	RKF45	-	-	0.0	0.256(-4)
"	8	"	-	3	-	RKF45	-	-	0.05	0.966(-2)
"	9	Neumann	4	2	no	DBCONF	no	no	0.0	0.172(-1)
"	10	"	4	2	no	DBCONF	no	no	0.05	0.157(-1)
"	11	Mixed	4	2	no	DBCONF	no	no	0.0	0.432(-2)
"	12	"	4	2	no	DBCONF	no	no	0.05	0.869(-2)
Short Cylinder	13	Dirichlet	-	3	-	RKF45	-	-	0.0	0.629(-2)
"	14	"	-	3	-	RKF45	-	-	0.05	0.147(-1)
"	15	Neumann	6	2	no	SIGMA	no	no	0.0	0.304(-1)
"	16	"	6	2	no	SIGMA	yes	yes	0.05	0.626(-1)
"	17(*)	Mixed	6	2	no	SIGMA	yes	yes	0.0	0.306(-1)
"	18(*)	"	6	2	no	SIGMA	yes	yes	0.05	0.468(-1)
Long Cylinder	19(*)	Dirichlet	-	3	-	RKF45	-	-	0.0	0.302(-2)
"	20(*)	"	-	3	-	RKF45	-	-	0.05	0.369(-2)
"	21	Neumann	6	2	3	SIGMA	yes	yes	0.0	0.543(-1)
"	22	"	6	2	3	SIGMA	yes	yes	0.05	0.482(-1)
"	23	Mixed	6	2	no	SIGMA	yes	yes	0.0	0.415(-1)
"	24	"	6	2	no	SIGMA	yes	yes	0.05	0.425(-1)

or the relative L^∞ error, that is:

$$E_{max} = \frac{\max\{|f(0,0) - f_c(0,0)|, |f(\pi,0) - f_c(\pi,0)|, |f(\theta_i, \phi_j) - f_c(\theta_i, \phi_j)|, \substack{i=1,2,\dots,35 \\ j=0,1,\dots,35}\}}{\max\{|f(0,0)|, |f(\pi,0)|, |f(\theta_i, \phi_j)|, \substack{i=1,2,\dots,35 \\ j=0,1,\dots,35}\}} \quad (5.17)$$

Let us define the sets:

$$A_1 = \{(\theta_i, 0) \mid \theta_i = i\frac{\pi}{10}, i = 0, 1, \dots, 10\} \quad (5.18)$$

$$A_2 = \{(\theta_i, \phi_j) \mid \theta_i = i\frac{\pi}{10}, i = 1, \dots, 9; \phi_j = j\frac{\pi}{4}, j = 0, 1, \dots, 8\} \cup \{0,0\} \cup \{\pi,0\} \quad (5.19)$$

$$A_3 = \{(\theta_i, \phi_j) \mid \theta_i = i\frac{\pi}{8}, i = 1, \dots, 7; \phi_j = j\frac{\pi}{4}, j = 0, 1, \dots, 6\} \cup \{0,0\} \cup \{\pi,0\} \quad (5.20)$$

Table 5.2 Axially symmetric non-convex obstacles

$$L_{max} = L_j = 8; \Omega_1 = A_1; \Omega_3 = A_2$$

Object	Recon- struction	Boundary condition	L	k_1	k_2	Reconstruction method	Penalization term North PoleEquator	ϵ	E_{L^2}
Vogel's Peanut	1(*)	Dirichlet	4	3	no	SIGMA	no no	0.0	0.128(-1)
"	2(*)	"	4	3	no	SIGMA	no no	0.05	0.175(-1)
"	3	Neumann	4	2	no	SIGMA	no no	0.0	0.596(-1)
"	4	"	4	2	no	SIGMA	no no	0.05	0.176(-1)
"	5	Mixed	4	2	no	SIGMA	no no	0.0	0.273(-1)
"	6	"	4	2	no	SIGMA	no no	0.05	0.128(-1)
Horizontal Platelet	7	Dirichlet	4	3	no	SIGMA	no no	0.0	0.451(-1)
"	8	"	4	3	no	SIGMA	no no	0.05	failure
"	9	Neumann	4	2	no	SIGMA	no no	0.0	0.103(+0)
"	10	"	4	2	no	SIGMA	no no	0.05	0.202(+0)
"	11(*)	Mixed	4	2	no	SIGMA	no no	0.0	0.861(-1)
"	12(*)	"	4	2	no	SIGMA	no no	0.05	0.156(0)
Vertical Peanut	13	Dirichlet	4	3	no	SIGMA	yes yes	0.0	0.172(-1)
"	14	"	4	3	no	SIGMA	yes yes	0.05	failure
"	15	Neumann	4	2	3	SIGMA	no no	0.0	0.725(-1)
"	16	"	4	2	3	SIGMA	no no	0.05	failure
"	17	Mixed	4	2	3	SIGMA	no no	0.0	0.649(-1)
"	18	"	4	2	3	SIGMA	no no	0.05	failure
Pseudo Apollo	19	Dirichlet	4	3	no	SIGMA	no no	0.0	0.602(-1)
"	20	"	4	3	no	SIGMA	no no	0.05	0.114(+0)
"	21(*)	Neumann	4	2	3	SIGMA	yes no	0.0	0.359(-1)
"	22(*)	"	4	2	3	SIGMA	yes no	0.05	0.325(-1)
"	23	Mixed	4	2	3	SIGMA	yes no	0.0	0.551(-1)
"	24	"	4	2	3	SIGMA	yes no	0.05	0.321(-1)

In our numerical experience we choose

$$\Omega_1 = A_1 \quad (5.21)$$

when the obstacles of Tables 5.1, 5.2 are considered or

$$\Omega_1 = A_3 \quad (5.22)$$

when the obstacles of Table 5.3 are considered. The set Ω_2 is either $\{2\}$, $\{3\}$, or $\{2,3\}$ (see Tables). Finally when the obstacles of Tables 5.1, 5.2 are considered Ω_3 is given by A_2 and when the obstacles of Tables 5.3 are considered Ω_3 is given by A_3 . We observe that with these choices the resonance condition (1.9) is satisfied. The far field data corresponding to these choices are obtained by solving numerically the corresponding direct problems that is the boundary value problems (1.3), (1.4), (1.5) or (1.3), (1.4), (1.6) or (1.3), (1.4), (1.7) using a T -matrix approach [16].

Table 5.3 Generic obstacles

Reconstruction method = "DUNLSJ", without penalization term,
 $L_{max} = L_g = 6$; $L_p = 4$; $k_1 = 2$; $\Omega_1 = A_3$; $\Omega_3 = A_3$

Object	Reconstruction	Boundary condition	ϵ	E_{L^2}
Corrugated Ellipsoid	1	Dirichlet	0.0	0.276(-1)
"	2	"	0.02	0.428(-1)
"	3(*)	Neumann	0.0	0.287(-1)
"	4(*)	"	0.02	0.417(-1)
"	5	Mixed	0.0	0.325(-1)
"	6	"	0.02	0.133(+0)
Ellipsoid	7	Dirichlet	0.0	0.291(-1)
"	8	"	0.02	0.296(-1)
"	9	Neumann	0.0	0.303(-1)
"	10	"	0.02	0.308(-1)
"	11	Mixed	0.0	0.152(+0)
"	12	"	0.02	0.245(-1)
Corrugated Cylinder	13	Dirichlet	0.0	0.534(-1)
"	14	"	0.02	0.652(-1)
"	15	Neumann	0.0	0.502(-1)
"	16	"	0.02	0.118(+0)
"	17(*)	Mixed	0.0	0.569(-1)
"	18(*)	"	0.02	0.675(-1)
Corrugated Platelet	19(*)	Dirichlet	0.0	0.663(-1)
"	20(*)	"	0.02	0.129(+0)
"	21	Neumann	0.0	0.105(+0)
"	22	"	0.02	failure
"	23	Mixed	0.0	0.651(-1)
"	24	"	0.02	0.175(+0)

Table 5.4 Performance as a function of ϵ

Object: Long Cylinder; Boundary condition: Dirichlet;

$L_{max} = L_g = 8$; $L_p = 6$; $k_1 = 3$; $\Omega_1 = A_1$ $\Omega_3 = A_3$

Reconstruction method= "SIGMA" without penalization term.

Reconstruction	ϵ	E_{max}	E_{L^2}
1	0.0	0.611(-1)	0.286(-1)
2	0.05	0.614(-1)	0.294(-1)
3	0.10	0.587(-1)	0.298(-1)
4	0.20	0.525(-1)	0.398(-1)
5	0.30	0.865(-1)	0.548(-1)
6	0.40	0.109(+0)	0.655(-1)
7	0.50	0.123(+0)	0.714(-1)
8	0.60	0.131(+0)	0.752(-1)

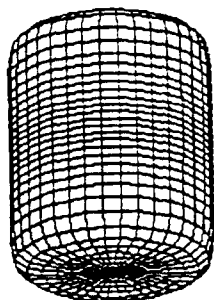
To the data $F_{i,i_1,j} = F_0(\hat{x}_j, k_{i_1}, \alpha_i)$ is added a random error term, that is $F_{i,i_1,j}$ is substituted with

$$F_{i,i_1,j}^* = F_{i,i_1,j} + \epsilon \zeta |F_{i,i_1,j}| \quad (5.23)$$

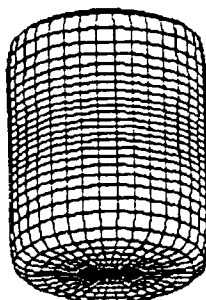
where $\epsilon \geq 0$ is a parameter and ζ is a random number uniformly distributed in $[-1, 1]$.

In our numerical experience we have $L_{max} = L_g = 6$ or 8 , $L_p = 4$ or 6 (see Tables). Finally in Table 5.4 we show the performance of our algorithms for increasing values of ϵ in the case of the acoustically soft long cylinder. The method based on the global minimization algorithm SIGMA appears to be the most powerful one at the price of higher computational cost. The computations previously described have been performed on a VAX 6310 with VMS Operating System.

ORIGINAL



RECONSTRUCTION 19
of table 5.1



RECONSTRUCTION 20
of table 5.1

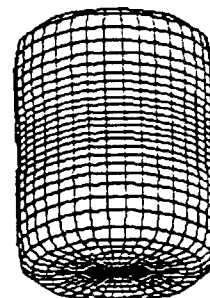
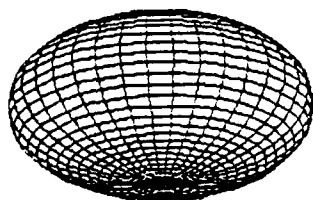
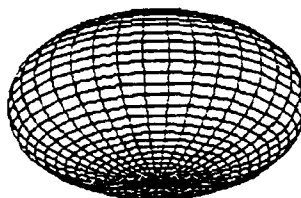


Fig. 5.1 *Long Cylinder*

ORIGINAL



RECONSTRUCTION 3
of table 5.1



RECONSTRUCTION 4
of table 5.1

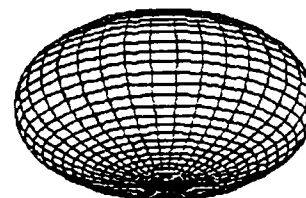
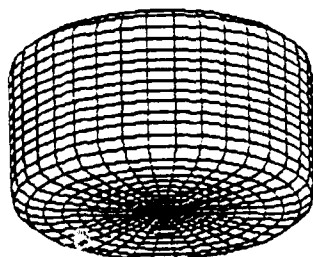
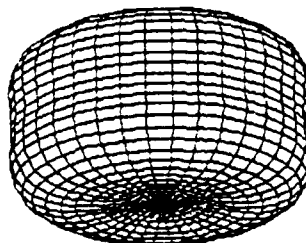


Fig. 5.2 *Oblate Ellipsoid*

ORIGINAL



RECONSTRUCTION 17
of table 5.1



RECONSTRUCTION 18
of table 5.1

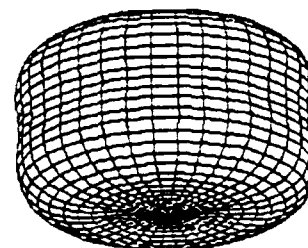
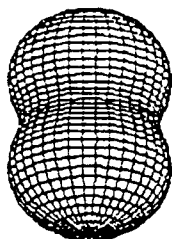
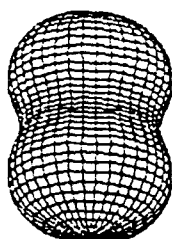


Fig. 5.3 *Short Cylinder*

ORIGINAL



RECONSTRUCTION 1
of table 5.2



RECONSTRUCTION 2
of table 5.2

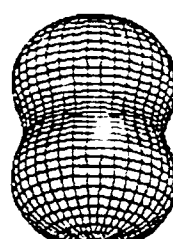
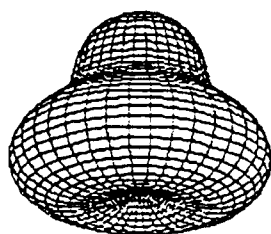
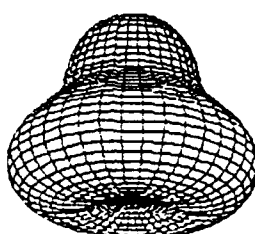


Fig. 5.4 *Vogel's Peanut*

ORIGINAL



RECONSTRUCTION 21
of table 5.2



RECONSTRUCTION 22
of table 5.2

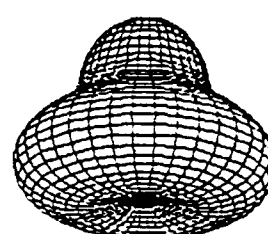
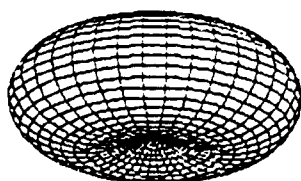
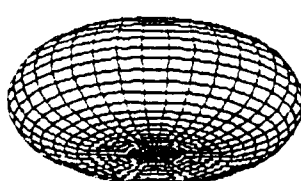


Fig. 5.5 *Pseudo Apollo*

ORIGINAL



RECONSTRUCTION 11
of table 5.2



RECONSTRUCTION 12
of table 5.2

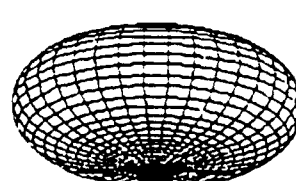


Fig. 5.6 *Horizontal Platelet*

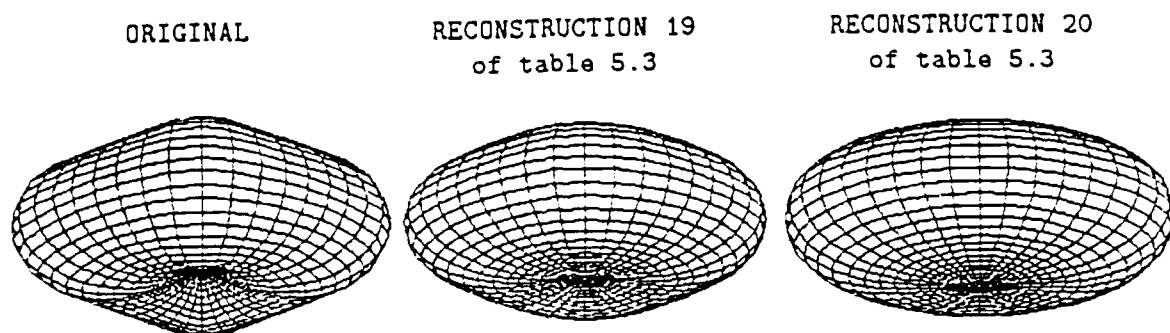


Fig. 5.7 *Corrugated Platelet*

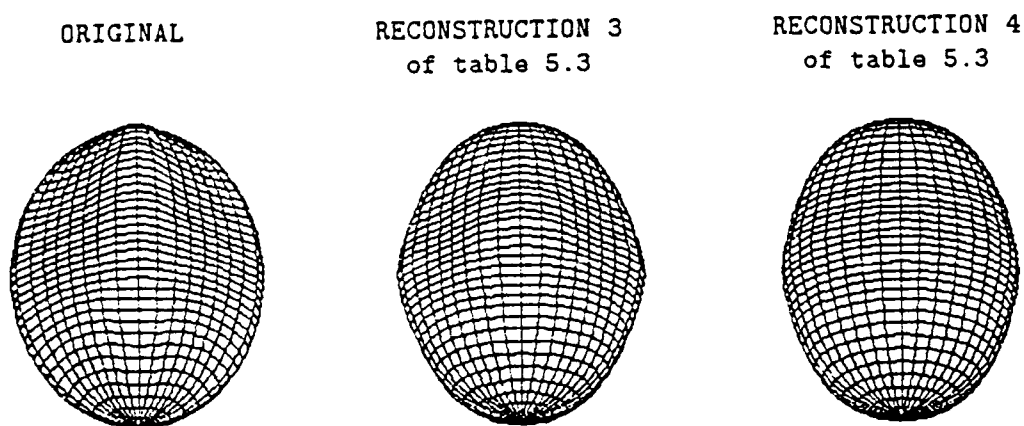


Fig. 5.8 *Corrugated Ellipsoid*

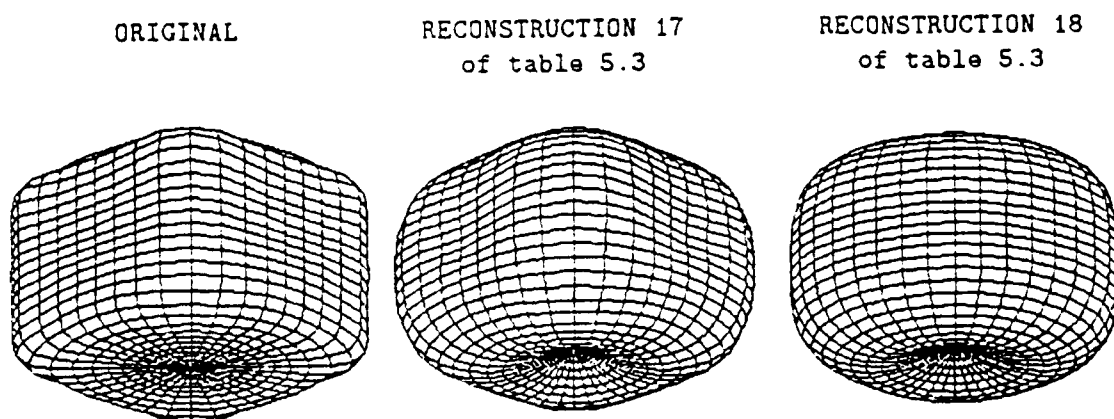


Fig. 5.9 *Corrugated Cylinder*

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5 APPENDIX 3

L. Misici, F. Zirilli: "Three dimensional time harmonic acoustic and electromagnetic inverse scattering in the resonance region", in Computational and Applied Mathematics, II, W.F. Ames and P.J. van der Houwen (Editors), Elsevier Science Publishers B.V. (North Holland), 1992, 335-344.

Three dimensional time harmonic acoustic and electromagnetic inverse scattering in the resonance region†

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Abstract

A numerical method for the three dimensional inverse acoustic and electromagnetic time harmonic scattering problem is presented. The far field patterns of the Helmholtz or vector Helmholtz equations generated by a known wave incident on an obstacle D are measured. These measurements are repeated for several incoming waves. From these measurements the boundary ∂D of the obstacle is reconstructed. The reconstruction procedure proposed here generalizes the "Herglotz function method" introduced by Colton and Monk [1] in the acoustic problem and is effective in the so called resonance region.

1. INTRODUCTION

Let \mathbb{R}^3 be the three dimensional euclidean space, $\underline{x} = (x, y, z) \in \mathbb{R}^3$ be a generic vector, (\cdot, \cdot) will denote the euclidean scalar product and $\|\cdot\|$ the euclidean norm. Let $D \subset \mathbb{R}^3$ be a bounded simply connected domain with smooth boundary ∂D that contains the origin. Let $u^i(\underline{x})$ be an incoming acoustic plane wave, that is:

$$u^i(\underline{x}) = e^{ik(\underline{x} \cdot \underline{\alpha})} \quad (1.1)$$

where $k > 0$ is the wave number and $\underline{\alpha} \in \mathbb{R}^3$ is a fixed unit vector. Let us denote with $u^s(\underline{x})$ the acoustic field scattered by the obstacle D and with $u(\underline{x})$ the total acoustic field, that is:

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$$u(\underline{x}) = u^i(\underline{x}) + u^s(\underline{x}) \quad (1.2)$$

The total acoustic field $u(\underline{x})$ satisfies the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus D \quad (1.3)$$

and the scattered acoustic field $u^s(\underline{x})$ satisfies the Sommerfeld radiation condition at infinity, that is:

$$\lim_{\|\underline{x}\| \rightarrow \infty} \|\underline{x}\| \left\{ \frac{\partial u^s}{\partial \|\underline{x}\|} - i k u^s \right\} = 0 \quad (1.4)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator. Moreover the total acoustic field $u(\underline{x})$ satisfies a boundary condition on ∂D . This boundary condition can be formulated in several different ways, depending on the nature of the obstacle D .

In [2],[3],[4] we have considered the acoustically soft obstacles that are characterized by the Dirichlet boundary condition:

$$u = 0 \quad \text{on } \partial D \quad (1.5)$$

In [4],[5] we have considered the acoustically hard obstacles characterized by the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \quad (1.6)$$

where ν is the unit normal on ∂D , and the obstacles characterized by an acoustic impedance that satisfy the mixed boundary condition

$$u + \chi \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \quad (1.7)$$

We assume that χ is a given constant. We consider three boundary value problems: the Dirichlet boundary value problem given by (1.3),(1.4),(1.5), the Neumann boundary value problem given by (1.3), (1.4),(1.6) and the mixed boundary value problem given by (1.3), (1.4),(1.7). In [6] it is shown that the scattered field $u^s(\underline{x})$ of the Dirichlet, the Neumann and mixed boundary value problem has the following expansion

$$u^s(\underline{x}) = \frac{e^{ik\|\underline{x}\|}}{\|\underline{x}\|} F_o(\hat{\underline{x}}, k, \underline{\alpha}) + O\left(\frac{1}{\|\underline{x}\|^2}\right) \quad \text{when } \|\underline{x}\| \rightarrow \infty \quad (1.8)$$

where $\hat{\underline{x}} = \frac{\underline{x}}{\|\underline{x}\|}$, $\underline{x} \neq 0$ and $F_o(\hat{\underline{x}}, k, \underline{\alpha})$ is the far field pattern generated by the incoming wave (1.1) that hits the obstacle D .

In [2],[3],[4],[5] we have introduced a numerical method for an inverse problem for the

three dimensional Helmholtz equation, that is from the knowledge of the nature of the obstacle i.e. the boundary condition on ∂D satisfied by $u(\underline{x})$ and from the far fields F_0 generated by several incoming waves we want to recover the shape of the obstacle ∂D . To be more precise let λ_n , $n = 1, 2, \dots$ be the eigenvalues of the Laplace operator in the interior of D , with Dirichlet boundary condition (1.5) or with Neumann boundary condition (1.6) or with the mixed boundary condition (1.7); let $B = \{\underline{x} \in \mathbb{R}^3 \mid \|\underline{x}\| < 1\}$, and ∂B be the boundary of B . We will consider the following inverse problem:

Problem 1.1 Inverse acoustic problem. Let us assume that $u(\underline{x})$ satisfies the Dirichlet boundary condition (1.5) or the Neumann boundary condition (1.6) or the mixed boundary condition (1.7). Let $\Omega_1 \subseteq \partial B$, $\Omega_2 \subset \{\xi \in \mathbb{R} \mid \xi > 0\}$ be two given sets such that $\lambda_i \notin \Omega_2$ $i = 1, 2, \dots$. From the knowledge of $F_0(\underline{\hat{x}}, k, \underline{\alpha})$, for $\underline{\alpha} \in \Omega_1$, $-k^2 \in \Omega_2$ determine the boundary of the obstacle ∂D .

We note that the condition $\lambda_i \notin \Omega_2$ $i = 1, 2, \dots$ is a non-resonance condition, that Ω_1 is the set of the directions of the incoming waves and that the far field F_0 is observed for $\underline{\hat{x}} \in \partial B$. Let us now consider the electromagnetic problem. For time harmonic waves the Maxwell equations are reduced to the time harmonic Maxwell equations [6] (see chapter 4).

In the following we will use occasionally complex vectors abusing of the notations.

Let $\underline{E}'(\underline{x})$ be the electric field associated to a linearly polarized time harmonic incoming wave propagating in an homogeneous isotropic medium, that is:

$$\underline{E}'(\underline{x}) = \underline{w} e^{ik(\underline{x}, \underline{\alpha})} \quad (1.9)$$

where $\underline{w}, \underline{\alpha} \in \mathbb{R}^3$ with $\|\underline{\alpha}\| = 1$ are given and $k > 0$ is the wave number, moreover we assume that:

$$\text{div} \underline{E}'(\underline{x}) = ik(\underline{w}, \underline{\alpha}) e^{ik(\underline{x}, \underline{\alpha})} = 0 \quad (1.10)$$

where $\underline{E}'(\underline{x}) = (E'_x(\underline{x}), E'_y(\underline{x}), E'_z(\underline{x}))$ and $\text{div} \underline{E}'(\underline{x}) = \frac{\partial E'_x(\underline{x})}{\partial x} + \frac{\partial E'_y(\underline{x})}{\partial y} + \frac{\partial E'_z(\underline{x})}{\partial z}$. We note that \underline{w} is the polarization vector and $\underline{\alpha}$ is the propagation direction of the incoming electric field. We note that the magnetic field $\underline{H}'(\underline{x})$ associated to this incoming wave is given by:

$$\underline{H}'(\underline{x}) = \frac{1}{ik} \text{curl} \underline{E}'(\underline{x}) \quad (1.11)$$

where $\text{curl} \underline{E}'(\underline{x}) = \left(\frac{\partial E'_z(\underline{x})}{\partial y} - \frac{\partial E'_y(\underline{x})}{\partial z}, \frac{\partial E'_x(\underline{x})}{\partial z} - \frac{\partial E'_z(\underline{x})}{\partial x}, \frac{\partial E'_y(\underline{x})}{\partial x} - \frac{\partial E'_x(\underline{x})}{\partial y} \right)$.

Let us denote with $\underline{E}^s(\underline{x})$ the electric field scattered by the obstacle D when hit by the incoming wave $\underline{E}'(\underline{x})$ and with

$$\underline{E}(\underline{x}) = \underline{E}'(\underline{x}) + \underline{E}^s(\underline{x}) \quad (1.12)$$

the total electric field. It is easy to see [6] (see chapter 4) that the time harmonic Maxwell

equations in an homogeneous isotropic medium that does not contain electric charge reduce to the vector Helmholtz equation, so that the scattered field $\underline{E}^s(\underline{x})$ satisfies:

$$\Delta \underline{E}^s(\underline{x}) + k^2 \underline{E}^s(\underline{x}) = 0 \quad \text{in } \mathbb{R}^3 \setminus D \quad (1.13)$$

together with the equation:

$$\operatorname{div} \underline{E}^s(\underline{x}) = 0 \quad \text{in } \mathbb{R}^3 \setminus D \quad (1.14)$$

while the Silver-Müller radiation condition at infinity [6] (see paragraph 4.2) reduces to:

$$\operatorname{curl} \underline{E}^s(\underline{x}) \times \underline{\hat{x}} - ik \underline{E}^s(\underline{x}) = o\left(\frac{1}{\|\underline{x}\|}\right), \quad \|\underline{x}\| \rightarrow \infty \quad (1.15)$$

where \times is the vector product. Let $\underline{\nu}(\underline{x})$ be the exterior unit normal to ∂D , for a perfectly conducting obstacle D , we will assume the following boundary condition:

$$\underline{E}(\underline{x}) \times \underline{\nu}(\underline{x}) = 0, \quad \underline{x} \in \partial D \quad (1.16)$$

In [6] it is shown that $\underline{E}^s(\underline{x})$, solution of the boundary value problem (1.13), (1.14), (1.15), (1.16) has the following expansion:

$$\underline{E}^s(\underline{x}) = \frac{e^{ik\|\underline{x}\|}}{\|\underline{x}\|} \underline{E}_0(\underline{\hat{x}}, k, \underline{\alpha}, \underline{w}) + O\left(\frac{1}{\|\underline{x}\|^2}\right), \quad \|\underline{x}\| \rightarrow \infty \quad (1.17)$$

where $\underline{E}_0(\underline{\hat{x}}, k, \underline{\alpha}, \underline{w})$ is the (electric) far field pattern generated by the incoming wave (1.9) that hits the obstacle D .

In [7] we introduce a numerical method for an inverse problem for the three dimensional vector Helmholtz equation (1.13). That is, from the knowledge of the nature of the obstacle (i.e. the fact that the obstacle is perfectly conducting) and from the (electric) far fields $\underline{E}_0(\underline{\hat{x}}, k, \underline{\alpha}, \underline{w})$ generated by several (known) incoming waves we want to recover the shape of the obstacle ∂D .

To be more precise let λ_n , $n = 1, 2, \dots$ be the eigenvalues of the "vector" Laplace operator restricted to the divergence free vector fields with the homogeneous boundary condition (1.16) in the interior of D . In [7] we have considered the following inverse problem:

Problem 1.2 Inverse electromagnetic problem. Let $\tilde{\Omega}_1 \subseteq \partial B$, $\tilde{\Omega}_2 \subset \mathbb{R}^3$, $\tilde{\Omega}_3 \subset \mathbb{R}$ be three given sets such that $\tilde{\lambda}_i \notin \tilde{\Omega}_3$, $i = 1, 2, \dots$ and let $\underline{E}_0(\underline{\hat{x}}, k, \underline{\alpha}, \underline{w})$ be the (electric) far field defined in (1.17). From the knowledge of $\underline{E}_0(\underline{\hat{x}}, k, \underline{\alpha}, \underline{w})$ for $\underline{\alpha} \in \tilde{\Omega}_1$, $\underline{w} \in \tilde{\Omega}_2$, $-k^2 \in \tilde{\Omega}_3$ determine the boundary of the obstacle ∂D .

We note that the condition $\tilde{\lambda}_i \notin \tilde{\Omega}_3$, $i = 1, 2, \dots$ is a non resonance condition, that $\tilde{\Omega}_1$ is the set of the directions of the incoming waves and that $\tilde{\Omega}_2$ is the set of the polarization vectors of the incoming waves.

The inverse Problems 1.1, 1.2 have been studied generalizing the "Herglotz function method" introduced by Colton and Monk [1]. This technique is supposed to be particularly effective in the resonance region, that is, when

$$kL \simeq 1 \quad (1.18)$$

where L is a characteristic length of the obstacle D .

In section 2 we develop the mathematical relations needed to develop our procedure and we outline the numerical method derived from them. Finally in section 3 we present some numerical results.

2. THE MATHEMATICAL FORMULATION OF THE INVERSE PROBLEM

We will restrict our attention here to the inverse Problem 1.1 associated to the Dirichlet boundary value problem (1.3), (1.4), (1.5). The remaining inverse problems are treated in a similar way and we refer to [3], [4], [5], [7].

For $\underline{x}, \underline{y} \in \mathbb{R}^3$ let

$$\Phi(k\|\underline{x} - \underline{y}\|) = \frac{e^{ik\|\underline{x} - \underline{y}\|}}{\|\underline{x} - \underline{y}\|} \quad (2.1)$$

be the Green's function of the Helmholtz operator with the Sommerfeld radiation condition at infinity. It is easy to see that:

$$\Phi(k\|\underline{x} - \underline{y}\|) = \frac{e^{ik\|\underline{x}\|}}{\|\underline{x}\|} e^{-ik(\underline{x}, \underline{y})} + O\left(\frac{1}{\|\underline{x}\|^2}\right) \quad \text{when } \|\underline{x}\| \rightarrow \infty \quad (2.2)$$

moreover from the Helmholtz formula [6] we have:

$$\frac{1}{4\pi} \int_{\partial D} (u(\underline{y}) \frac{\partial \Phi}{\partial \nu(\underline{y})} - \Phi \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})}) d\sigma_1(\underline{y}) = \begin{cases} -u'(\underline{x}) & \text{if } \underline{x} \in D \\ u'(\underline{x}) & \text{if } \underline{x} \in \mathbb{R}^3 \setminus D \end{cases} \quad (2.3)$$

where $d\sigma_1(\underline{y})$ is the surface measure on ∂D .

Substituting (2.2) in (2.3) and using (1.8) we have:

$$F_o(\underline{\hat{x}}, k, \underline{\alpha}) = \frac{1}{4\pi} \int_{\partial D} (u(\underline{y}) \frac{\partial e^{-ik(\underline{\hat{x}}, \underline{y})}}{\partial \nu(\underline{y})} - \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} e^{-ik(\underline{\hat{x}}, \underline{y})}) d\sigma_1(\underline{y}) \quad (2.4)$$

Let $g(\underline{\hat{x}}) \in L^2(\partial B, d\sigma_2)$ where $L^2(\partial B, d\sigma_2)$ is the space of square integrable functions with respect to the surface measure on the unit sphere $d\sigma_2$ and $g(\underline{\hat{x}})$ is the complex conjugate of $g(\underline{\hat{x}})$.

For every $g(\underline{\hat{x}}) \in L^2(\partial B, d\sigma_2)$ from (2.4) interchanging the integrals we have:

$$\begin{aligned}
\int_{\partial B} F_o(\underline{\hat{x}}, k, \underline{\alpha}) \overline{g(\underline{\hat{x}})} d\sigma_2(\underline{\hat{x}}) &= \frac{1}{4\pi} \int_{\partial B} d\sigma_2(\underline{\hat{x}}) \overline{g(\underline{\hat{x}})} \\
&= \int_{\partial D} (u(\underline{y}) \frac{\partial e^{-ik(\underline{\hat{x}}, \underline{y})}}{\partial \nu(\underline{y})} - \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} e^{-ik(\underline{\hat{x}}, \underline{y})}) d\sigma_1(\underline{y}) = \\
&= \frac{1}{4\pi} \int_{\partial D} (u(\underline{y}) \frac{\partial v(k\underline{y})}{\partial \nu(\underline{y})} - \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} \overline{v(k\underline{y})}) d\sigma_1(\underline{y})
\end{aligned} \tag{2.5}$$

where

$$v(k\underline{y}) = \int_{\partial B} g(\underline{\hat{x}}) e^{ik(\underline{\hat{x}}, \underline{y})} d\sigma_2(\underline{\hat{x}}) \tag{2.6}$$

It is easy to see that $v(k\underline{y})$ satisfies the Helmholtz equation for $\underline{y} \in \mathbb{R}^3$, moreover $v(k\underline{y})$ is the Herglotz wave function corresponding to the Herglotz kernel $g(\underline{\hat{x}})$. Since the total acoustic field u satisfies the boundary condition (1.5) on the surface ∂D formula (2.5) reduces to:

$$\int_{\partial B} F_o(\underline{\hat{x}}, k, \underline{\alpha}) \overline{g(\underline{\hat{x}})} d\sigma_2(\underline{\hat{x}}) = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial u(\underline{y})}{\partial \nu(\underline{y})} \overline{v(k\underline{y})} d\sigma_1(\underline{y}) \tag{2.7}$$

We restrict our attention to the Dirichlet Herglotz domains, that is domains such that the unique solution v of:

$$(\Delta + k^2)v = 0 \quad \underline{x} \in D \tag{2.8}$$

$$v = -\frac{e^{-ik\|\underline{x}\|}}{k\|\underline{x}\|} \quad \underline{x} \in \partial D \tag{2.9}$$

is given by (2.6) for a suitable choice $g_H(\underline{\hat{x}}, k)$ of $g(\underline{\hat{x}})$

We note that in the definition of Herglotz domains we have exploited the hypothesis $-k^2 \neq \lambda_i$, $i = 1, 2, \dots$. A simple computation shows that the sphere with center the origin is an Herglotz domain that is the class of the Herglotz domains is not empty. In (2.7), let v be the Herglotz wave function associated to $g_H(\underline{\hat{x}}, k)$, using (2.9) and the Helmholtz formula (2.3) we have:

$$\int_{\partial B} F_o(\underline{\hat{x}}, k, \underline{\alpha}) \overline{g_H(\underline{\hat{x}}, k)} d\sigma_2(\underline{\hat{x}}) = \frac{1}{k} \tag{2.10}$$

when $g_H(\underline{\hat{x}}, k)$ is the Herglotz kernel, formula (2.10) holds $\forall k, \underline{\alpha}$. Problem 1.1 proposed in section 1 will be solved in three steps:

- (i) from the knowledge of some far fields F_o using (2.10) determine the Herglotz kernel $g_H(\underline{\hat{x}}, k)$ of the domain D
- (ii) from $g_H(\underline{\hat{x}}, k)$ using (2.6) find the corresponding Herglotz wave function v

(iii) determine ∂D using (2.9)

The steps (i),(ii),(iii) are performed with a fixed k , however when necessary several (two or three) values of k can be used [4],[5].

The numerical method that exploits the previously described reconstruction procedure consists of some linear algebra computations to perform step (i), explicit analytic computation to perform step (ii), a global minimization procedure to perform step (iii). The global minimization algorithm used is based on the numerical integration of a system of stochastic differential equations [9],[10]. Further details are contained in [2],[3],[4],[5],[7].

3. NUMERICAL RESULTS

The surfaces ∂D considered are the following ones:

$$1- \quad \text{Oblate Ellipsoid} \quad \left(\frac{2}{3}x\right)^2 + \left(\frac{2}{3}y\right)^2 + z^2 = 1 \quad (3.1)$$

$$2- \quad \text{Prolate Ellipsoid} \quad x^2 + y^2 + \left(\frac{2}{3}z\right)^2 = 1 \quad (3.2)$$

$$3- \quad \text{Vogel's Peanut} \quad r = \frac{2}{3}(\cos^2 \theta + \frac{1}{4} \sin^2 \theta)^{1/2} \quad (3.3)$$

$$4- \quad \text{Short Cylinder} \quad \left(\left(\frac{2}{3}x\right)^2 + \left(\frac{2}{3}y\right)^2\right)^5 + z^{10} = 1 \quad (3.4)$$

$$5- \quad \text{Pseudo Apollo} \quad r = \frac{3}{5}\left(\frac{17}{4} + 2 \cos 3\theta\right)^{1/2} \quad (3.5)$$

All these surfaces are cylindrically symmetric with respect to the z -axis and the surfaces 1,2,3,4 are also symmetric with respect to the equator.

We observe that the obstacles D corresponding to 1,2,4 are convex and the ones corresponding to 3,5 are not convex. Finally a characteristic length L of the obstacles can be chosen equal 1.

Let (r, θ, ϕ) be polar coordinates and $\hat{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, $\partial D = \{r = f(\theta, \phi) \mid 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}$ for $j = 0, 1, \dots, 36$ let $\theta_j = \frac{\pi j}{36}$, $f(\theta_j, 0)$ be the exact values of the surfaces given by (3.1), ..., (3.5) and $f_c(\theta_j, 0)$ be the values reconstructed performing the numerical procedure described in section 2. The relative L^2 error in the points $\{(\theta_j, 0) \mid j = 0, 1, \dots, 36\}$, that is :

$$E_{L^2} = \left[\frac{\sum_{j=0}^{36} (f(\theta_j, 0) - f_c(\theta_j, 0))^2}{\sum_{j=0}^{36} f^2(\theta_j, 0)} \right]^{\frac{1}{2}} \quad (3.6)$$

is used as a performance index.

In the results shown below we use 9-11 different directions for the incoming waves and in the case of the electromagnetic problem two linearly independent polarizations for each incoming direction. The far fields are obtained from the numerical solution of the corresponding boundary value problems. That is, for example, for the acoustic Dirichlet inverse problem the boundary value problem (1.3),(1.4),(1.5). A random error of order 1%-5% is added to the numerically computed far fields in order to simulate actual measurements. The far fields are supposed to be known on the full solid angle (complete data) or only on a finite set (9-11) directions (incomplete data).

The results obtained with our reconstruction procedure are shown in tables 3.1, 3.2, 3.3, 3.4.

Table 3.1
Acoustic Dirichlet inverse problem

Object	k	Number of incoming waves	Complete data	E_{L_2}
Oblate Ellipsoid	3	9	no	0.0071
Prolate Ellipsoid	3	9	no	0.031
Vogel's Peanut	3	9	yes	0.019
Short Cylinder	3	9	no	0.0091
Pseudo Apollo	4	9	yes	0.062

Table 3.2
Acoustic Neumann inverse problem

Object	k	Number of incoming waves	Complete data	E_{L_2}
Oblate Ellipsoid	3	9	no	0.0079
Prolate Ellipsoid	3	9	no	0.0139
Vogel's Peanut	3	9	no	0.0462
Short Cylinder	2	9	yes	0.0476
Pseudo Apollo	2-3	9	yes	0.0353

Table 3.3
Acoustic Mixed inverse problem ($\chi = 1$)

Object	k	Number of incoming waves	Complete data	E_{L_2}
Oblate Ellipsoid	4	9	yes	0.0023
Prolate Ellipsoid	4	9	yes	0.017
Vogel's Peanut	3	9	no	0.027
Short Cylinder	3	9	yes	0.014
Pseudo Apollo	2-3	9	yes	0.036

Table 3.4
Electromagnetic inverse problem

Object	k	Number of incoming waves	Complete data	E_{L_2}
Oblate Ellipsoid	3	18	yes	0.0304
Prolate Ellipsoid	3	18	yes	0.0323
Vogel's Peanut	3	18	yes	0.0264
Short Cylinder	2.5	22	yes	0.0521
Pseudo Apollo	4	22	yes	0.0411

As an example in the figures 1 and 2 the quality of our reconstructions is shown.

ORIGINAL

RECONSTRUCTED

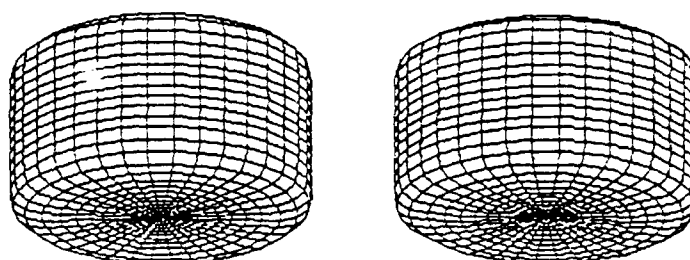


fig. 1

Short Cylinder

Reconstruction of Table 3.1

ORIGINAL

RECONSTRUCTED

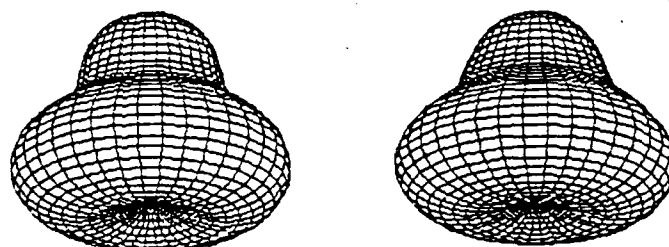


fig. 2

Pseudo Apollo

Reconstruction of Table 3.4

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